

Generalized stacked contact process with variable host fitness

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Abstract The stacked contact process is a three-state spin system that describes the co-evolution of a population of hosts together with their symbionts. In a nutshell, the hosts evolve according to a contact process while the symbionts evolve according to a contact process on the dynamic subset of the lattice occupied by the host population, indicating that the symbiont can only live within a host. This paper is concerned with a generalization of this system in which the symbionts may affect the fitness of the hosts by either decreasing (pathogen) or increasing (mutualist) their birth rate. Our main results show the existence of a parameter region in which hosts and symbionts, in this case pathogens, coexist in spatial dimensions $d > 1$ whereas only the host survives on the one-dimensional integer lattice. The mean-field approximation of the process is also studied in detail and compared to the spatial model.

1. Introduction

The stochastic model considered in this paper is a generalization of the stacked contact process introduced in [3] and studied analytically in [10]. The stacked contact process is a spatial stochastic process based on the framework of interacting particle systems that describes the co-evolution of a population of hosts together with their symbionts. Individuals are located on the d -dimensional integer lattice and interact with their nearest neighbors. The model assumes that the symbionts can only live in association with their host (obligate relationship) and are transmitted both vertically from associated hosts to their offspring and horizontally from associated hosts to nearby unassociated hosts. The stacked contact process [3, 10] also assumes that all the hosts give birth and die at the same rate regardless of whether they are associated with a symbiont or not, meaning that the symbionts have no effect on the fitness of their host.

In contrast, this paper considers the natural generalization of the stacked contact process in which associated and unassociated hosts have different birth rates: symbionts that increase the birth rate of their host, and therefore have a beneficial effect, are referred to as mutualists, whereas symbionts that decrease the birth rate of their host, and therefore have a detrimental effect, are referred to as pathogens. Formally, the state of the system at time t is a spatial configuration

$$\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$$

where state 0 means empty, state 1 means occupied by an unassociated host, and state 2 means occupied by a host associated with a symbiont. Letting

$$f_i(x, \xi) := (1/2d) \text{ card } \{y \in \mathbb{Z}^d : \sum_{j=1,2,\dots,d} |x_j - y_j| = 1 \text{ and } \xi(y) = i\}$$

*Research supported in part by an NSERC PGS D2 Award

†Research supported in part by NSA Grant MPS-14-040958.

AMS 2000 subject classifications: Primary 60K35

Keywords and phrases: Multitype contact process, forest fire model, host, pathogen, mutualist.

be the fraction of nearest neighbors of vertex x which are in state i , hosts and symbionts co-evolve according to the spin system whose transition rates at vertex x are given by

$$\begin{array}{llll} 0 \rightarrow 1 & \text{at rate } \lambda_{10} f_1(x, \xi) & 1 \rightarrow 0 & \text{at rate } 1 \\ 0 \rightarrow 2 & \text{at rate } \lambda_{20} f_2(x, \xi) & 2 \rightarrow 0 & \text{at rate } 1 \\ 1 \rightarrow 2 & \text{at rate } \lambda_{21} f_2(x, \xi) & 2 \rightarrow 1 & \text{at rate } \delta. \end{array} \quad (1)$$

The first four transition rates indicate that unassociated hosts give birth at rate λ_{10} , hosts associated with a symbiont give birth at rate λ_{20} and, regardless of whether they are associated or not, all the hosts die at the normalized rate one. An offspring produced at x is sent to a vertex chosen uniformly at random among the nearest neighbors but the birth is suppressed when the target site is already occupied, which models competition for space. The offspring is always of the same type as its parent, indicating that the symbiont is always transmitted vertically. The process described by these four transitions is the multitype contact process [12]. The effect of the symbiont on the host is modeled by the choice of the two birth rates: the symbiont is

a pathogen when $\lambda_{20} < \lambda_{10}$ and a mutualist when $\lambda_{20} > \lambda_{10}$.

The last two transition describe the symbiont dynamics within the host population. The symbiont spreads to adjacent unassociated hosts at rate λ_{21} , which corresponds to a horizontal transmission of the symbiont. Finally, hosts associated with a symbiont become unassociated at rate δ , which we simply call the recovery rate even when the symbiont is a mutualist.

The stacked contact process [3, 10] is obtained by setting $\lambda_{20} = \lambda_{10}$. This corresponds to the neutral case in which the symbionts have no effect on the fertility of their hosts, i.e., all the hosts have the same birth rate. The analysis of this special case in [10] is somewhat facilitated by the fact that the process is attractive and monotone with respect to its parameters. In contrast, in the general case when $\lambda_{10} \neq \lambda_{20}$, standard couplings to compare processes starting from different configurations or with different parameters fail to show attractiveness and monotonicity.

Mean-field approximation. Before studying the spatial stochastic process, we first look at its non-spatial deterministic counterpart called mean-field approximation. This new model is obtained by assuming that the population is well-mixing, which results in a system of ordinary differential equations for the density u_j of vertices in state $j = 0, 1, 2$. Since $u_0 + u_1 + u_2 = 1$, in order to study the mean-field model, it suffices to focus on two densities, say u_1 and u_2 . For the transition rates given in (1), this system consists of the following coupled equations:

$$\begin{aligned} u_1' &= \lambda_{10} u_0 u_1 - u_1 + \delta u_2 - \lambda_{21} u_1 u_2 \\ u_2' &= \lambda_{20} u_0 u_2 - u_2 - \delta u_2 + \lambda_{21} u_1 u_2. \end{aligned} \quad (2)$$

Clearly the feasible region

$$\Lambda := \{(u_1, u_2) \in \mathbb{R}^2 : u_1, u_2 \geq 0 \text{ and } u_1 + u_2 \leq 1\}$$

is forward invariant. In addition, the line segment $\Lambda \cap \{u_2 = 0\}$ is forward invariant, as well as the line segment $\Lambda \cap \{u_1 = 0\}$ when $\delta = 0$. Three boundary equilibria are possible:

1. $(0, 0)$,

2. $(1 - 1/\lambda_{10}, 0)$ when $\lambda_{10} > 1$ and
3. $(0, 1 - 1/\lambda_{20})$ when $\delta = 0$ and $\lambda_{20} > 1$.

To account for the additional invariant line segments, define

$$\Lambda_+ = \begin{cases} \{(u_1, u_2) \in \Lambda : \min(u_1, u_2) > 0\} & \text{if } \delta = 0 \\ \{(u_1, u_2) \in \Lambda : u_2 > 0\} & \text{if } \delta > 0. \end{cases}$$

Then we have the following (fairly complete) description of the dynamics.

Theorem 1 – *Under the following conditions, the stated equilibrium p is globally stable on Λ_+ .*

1. (extinction) $p = (0, 0)$, if $\max(\lambda_{10}, \lambda_{20}) \leq 1$ (here p is stable on all of Λ).
2. (coexistence) some unique p in the interior of Λ , if either
 - (a) $\delta > 0$, $\lambda_{10} \leq 1$ and $\lambda_{20} > 1 + \delta$,
 - (b) $\delta > 0$, $\lambda_{10} > 1$ and

$$\frac{\lambda_{20}}{\lambda_{10}} + \lambda_{21} \left(1 - \frac{1}{\lambda_{10}}\right) > 1 + \delta, \quad (3)$$

- (c) $\delta = 0$, $\lambda_{10} > 1$, inequality (3) holds and either $\lambda_{20} \leq 1$ or

$$\frac{\lambda_{10}}{\lambda_{20}} - \lambda_{21} \left(1 - \frac{1}{\lambda_{20}}\right) > 1. \quad (4)$$

3. (1s win) $p = (1 - 1/\lambda_{10}, 0)$, if $\delta = 0$, $\lambda_{10} > 1$, (3) does not hold and (4) holds.
4. (2s win) $p = (0, 1 - 1/\lambda_{20})$, if $\delta = 0$, $\lambda_{20} > 1$, (4) does not hold and (3) holds.

Some of these conditions can be intuitively understood as follows:

- If $\lambda_{10} > 1$, then inequality (3) says that the 2s can invade the 1s in equilibrium.
- If $\delta = 0$ and $\lambda_{20} > 1$, then inequality (4) says that the 1s can invade the 2s in equilibrium.

Theorem 1 is proved in Section 2.

Spatial stochastic process. As in the mean-field model, the spatial stochastic process can exhibit four regimes: either the host population survives or goes extinct and, in case of survival, the symbiont population either goes extinct, occupies some but not all hosts, or occupies all hosts. In addition, when the recovery rate $\delta > 0$, survival of associated hosts implies coexistence of associated and unassociated hosts. As pointed out previously, the stacked contact process obtained by setting $\lambda_{10} = \lambda_{20}$ has several nice properties such as attractiveness and monotonicity. In contrast, these properties do not seem to hold for our process, i.e., standard couplings fail to show these properties. Coupling can however be used to compare the process with other popular interacting particle systems and collect interesting results. We start by comparing the process with the basic and the multitype contact processes using simple coupling techniques. We also show that the process inherits some of the properties of the forest fire model though, because of the lack of monotonicity, this does not simply follow from a standard coupling argument.

In the limiting case when the recovery rate $\delta = \infty$, all the symbionts die instantaneously so the host dynamics reduces to the basic contact process

$$0 \rightarrow 1 \quad \text{at rate } \lambda_{10} f_1(x, \xi) \quad 1 \rightarrow 0 \quad \text{at rate } 1.$$

There exists a critical value $\lambda_c \in (0, \infty)$ such that above this critical value the host population survives whereas at and below this critical value the population goes extinct [1]. Assume from now on that the recovery rate is finite and let

$$\xi_t^1 := \mathbf{1}\{x \in \mathbb{Z}^d : \xi_t(x) \neq 0\} \quad \text{and} \quad \xi_t^2 := \mathbf{1}\{x \in \mathbb{Z}^d : \xi_t(x) = 2\}$$

be the process that keeps track of the hosts and the process that keeps track of the hosts associated to a symbionts, respectively. The transitions for the first process satisfy

$$\begin{aligned} 0 &\rightarrow 1 \quad \text{at rate at least} \quad \min(\lambda_{10}, \lambda_{20}) f_1(x, \xi^1) \\ 0 &\rightarrow 1 \quad \text{at rate at most} \quad \max(\lambda_{10}, \lambda_{20}) f_1(x, \xi^1) \end{aligned}$$

while $1 \rightarrow 0$ one. In particular, this process can be coupled with the basic contact process described above to deduce that, for all $x \in \mathbb{Z}^d$,

$$\begin{aligned} \liminf_{t \rightarrow \infty} P(\xi_t(x) \neq 0) &> 0 \quad \text{when} \quad \min(\lambda_{10}, \lambda_{20}) > \lambda_c \\ \lim_{t \rightarrow \infty} P(\xi_t(x) \neq 0) &= 0 \quad \text{when} \quad \max(\lambda_{10}, \lambda_{20}) \leq \lambda_c. \end{aligned} \tag{5}$$

This follows from Theorem III.1.5 in [11], which applies to general two-state spin systems, together with obvious inequalities relating the transition rates of our process and their counterpart for the basic contact process. Similarly, the transitions for the second process satisfy

$$\begin{aligned} 0 &\rightarrow 1 \quad \text{at rate at least} \quad \min(\lambda_{20}, \lambda_{21}) f_1(x, \xi^2) \\ 0 &\rightarrow 1 \quad \text{at rate at most} \quad \max(\lambda_{20}, \lambda_{21}) f_1(x, \xi^2) \end{aligned}$$

while $1 \rightarrow 0$ at rate $1 + \delta$, from which it follows that, for all $x \in \mathbb{Z}^d$,

$$\begin{aligned} \liminf_{t \rightarrow \infty} P(\xi_t(x) = 2) &> 0 \quad \text{when} \quad \min(\lambda_{20}, \lambda_{21}) > (1 + \delta) \lambda_c \\ \lim_{t \rightarrow \infty} P(\xi_t(x) = 2) &= 0 \quad \text{when} \quad \max(\lambda_{20}, \lambda_{21}) \leq (1 + \delta) \lambda_c. \end{aligned} \tag{6}$$

The four parameter regions in (5)–(6) are illustrated in the four diagrams of Figure 1. So far, the behavior of the stochastic process agrees with the behavior of the mean-field model described in Theorem 1, if we think of the mean-field model as having $\lambda_c = 1$ – note that $\min(\lambda_{20}, \lambda_{21}) > 1 + \delta$ implies (3) holds.

Setting $\lambda_{21} = \delta = 0$, the process reduces to the multitype contact process completely analyzed when the death rates are equal in [12]. The transition rates become

$$\begin{aligned} 0 &\rightarrow 1 \quad \text{at rate} \quad \lambda_{10} f_1(x, \xi) & 1 &\rightarrow 0 \quad \text{at rate} \quad 1 \\ 0 &\rightarrow 2 \quad \text{at rate} \quad \lambda_{20} f_2(x, \xi) & 2 &\rightarrow 0 \quad \text{at rate} \quad 1. \end{aligned}$$

In this case, the type with the larger birth rate outcompetes the other type provided its birth rate is also strictly larger than the critical value of the single-type contact process. This result has first been proved in [12] using duality techniques and again in [7] using also a block construction in order to prove that the long-term behavior of the process is not altered by small perturbations of the parameters. Using in addition a coupling argument, it can be easily deduced from [7, Propositions 3.1–3.2] that, the recovery rate δ being fixed, for all $x \in \mathbb{Z}^d$,

$$\lim_{t \rightarrow \infty} P(\xi_t(x) = 2) = 0 \quad \text{when} \quad \lambda_{10} > \lambda_{20} \text{ and } \lambda_{21} \text{ is small.} \tag{7}$$

The parameter region in (7) is shown in the bottom left diagram of Figure 1. Although it is not recorded in Theorem 1, this behavior also agrees with that of the mean-field model, which can be verified by an examination of the nullclines described in Section 2 – here the assumption that λ_{21} is small is important.

The forest fire model, also referred to as epidemics with recovery, is the three-state spin system with a cyclic dynamics described by the following three transitions:

$$\begin{aligned} 0 &\rightarrow 2 \quad \text{at rate } \alpha f_2(x, \xi) & 2 &\rightarrow 1 \quad \text{at rate } 1 \\ 1 &\rightarrow 0 \quad \text{at rate } \beta. \end{aligned}$$

The three states are interpreted as 0 = alive, 2 = on fire and 1 = burnt, but can also be thought respectively as healthy, infected and immune in the context of epidemics. This process has been studied in [6] which proves the existence of a critical value $\alpha_c \in (0, \infty)$ such that, in $d = 2$,

$$\liminf_{t \rightarrow \infty} P(\xi_t(x) = 2) > 0 \quad \text{when } \alpha > \alpha_c \text{ and } \beta > 0.$$

Because the dynamics is cyclic, basic couplings between the forest fire model and our process do not lead to any useful stochastic ordering between the two systems. However, the proof in [6] easily extends to our process in a certain parameter region. Indeed, in addition to general geometrical properties and percolation results which are not related to the specific dynamics of the forest fire model, the key estimates in [6] rely on the following two ingredients:

- (a) The set of burning trees dominates its counterpart in the process with no regrowth ($\beta = 0$) provided both processes start from the same configuration.
- (b) In regions that have not been on fire for at least S units of time, the set of trees which are alive dominates a product measure with density $1 - e^{-\beta S}$.

Now, fix $\delta \geq 0$, let $\beta = 1/(\delta + 1)$ and consider the spin system on the two-dimensional integer lattice whose dynamics is described by the five transitions

$$\begin{aligned} 0 &\rightarrow 2 \quad \text{at rate } \beta \lambda_{20} f_2(x, \xi) & 1 &\rightarrow 0 \quad \text{at rate } \beta \\ 1 &\rightarrow 2 \quad \text{at rate } \beta \lambda_{21} f_2(x, \xi) & 2 &\rightarrow 0 \quad \text{at rate } \beta \\ & & 2 &\rightarrow 1 \quad \text{at rate } \beta \delta. \end{aligned}$$

In this process, trees burn for an exponential amount of time with rate $\beta + \beta\delta = 1$ as in the original forest fire model. It follows that the domination property (a) remains true: the set of burning trees in this new process dominates its counterpart in the forest fire model with no regrowth and in which the fire spreads by contact at rate $\alpha = \beta \lambda_{20}$. Since the transition $1 \rightarrow 0$ again occurs spontaneously at rate β , the domination of the product measure (b) remains true as well. In particular,

$$\liminf_{t \rightarrow \infty} P(\xi_t(x) = 2) > 0 \quad \text{when } \lambda_{20} > (\delta + 1) \alpha_c.$$

This holds for all $\lambda_{21} \geq 0$. Since the proof in [6] is based on a block construction, which supports small perturbations of the system, we also obtain coexistence in the process (1) under the same assumptions and provided λ_{10} is sufficiently small. In conclusion, in $d = 2$,

$$\liminf_{t \rightarrow \infty} P(\xi_t(x) = 2) > 0 \quad \text{when } \lambda_{20} > (\delta + 1) \alpha_c \text{ and } \lambda_{10} \text{ is small.} \quad (8)$$

The parameter region in (8) is shown in the two diagrams on the right of Figure 1.

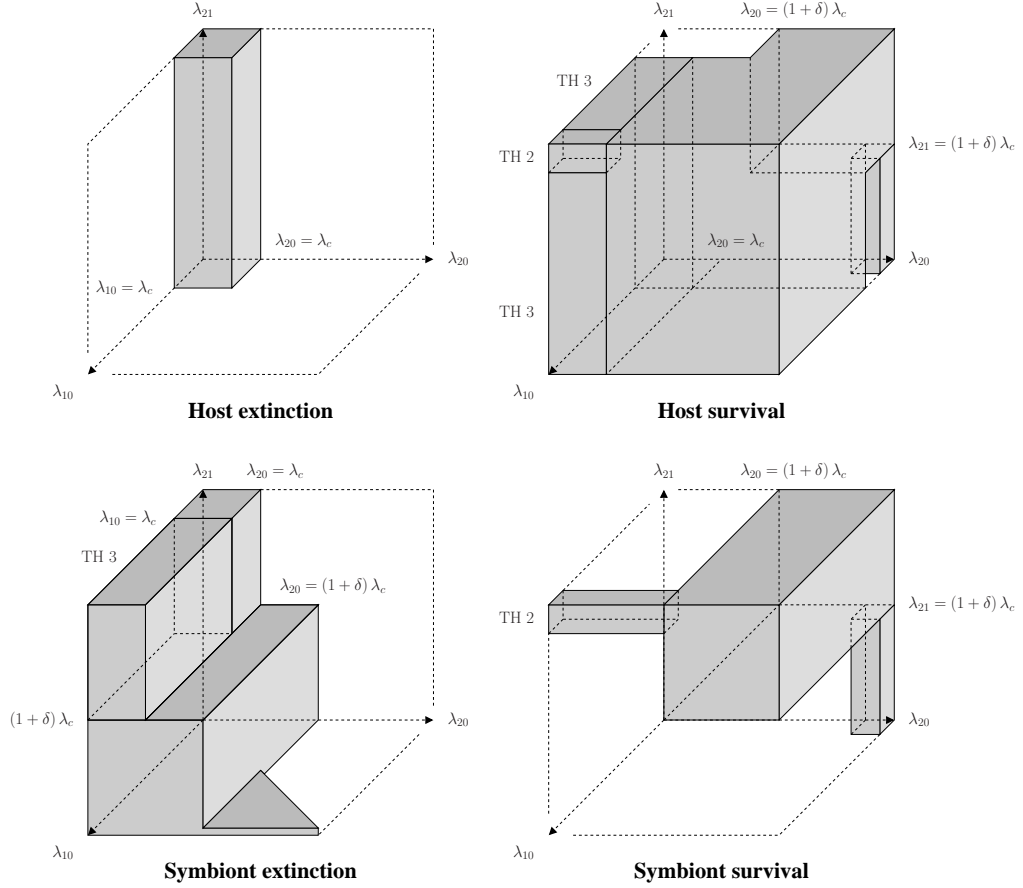


FIGURE 1. Parameter regions in which the host/symbiont dies out/survives. In the parameter region next to TH 2, the host and the symbiont coexist when $d > 1$ while in the parameter region next to TH 3, the host survives and the symbiont dies out when $d = 1$ and the recovery rate $\delta = 0$.

We now focus on the parameter region where $\lambda_{10} > \lambda_c > \lambda_{20}$ which is not covered by our previous results. In this case, the host population survives without the symbiont, which is a pathogen, and the main question is whether associated and unassociated hosts coexist. Theorem 1 says that in the mean-field model they do coexist, provided λ_{21} is large enough. Interestingly, our results show that for the stochastic process, the answer depends on the spatial dimension.

To begin with, we show that, even when infected hosts are sterile, coexistence is possible in spatial dimensions $d > 1$ but we also point out that our proof relies on geometric arguments that do not hold in one dimension. More precisely, we have the following theorem.

Theorem 2 – Assume that $\delta > 0$ and $d > 1$. Then, there is a large such that

$$\liminf_{t \rightarrow \infty} P(\xi_t(x) = 2) > 0 \text{ for all } x \in \mathbb{Z}^d \text{ when } \min(\lambda_{21}, \lambda_{10}/\lambda_{21}) \geq a.$$

Note that, when $\lambda_{20} > (1+\delta)\lambda_c$, the result directly follows from (6) so the theorem is only interesting when infected hosts have a low birth rate. The proof relies on a block construction, a technique introduced in [2]. Having a tiling of the lattice into large squares and letting S_1 and S_2

be two adjacent squares, the main ingredient is to prove that, whenever S_1 contains a large number of pathogens while S_2 does not have too many pathogens,

- there is a large connected component void of pathogens included in S_2 and
- the number of pathogens on the boundary of this connected component is large.

This is used together with the fact that $\delta > 0$ to show that, in a short time, one of the infected hosts along the boundary recovers. This recovery event is followed with probability close to one by a wave of births and then a wave of infections which, together, ensure that a positive fraction of the square S_2 becomes infected. Large deviations estimates are used to make this precise.

In contrast with Theorem 2, when $\delta = 0$ and $d = 1$, and even when the infection rate and the birth rate of healthy hosts are very large, the pathogen is unable to survive.

Theorem 3 – *Assume that $\lambda_{10} > \lambda_c > \lambda_{20}$ and $\delta = 0$ and $d = 1$. Then, starting from any configuration with infinitely many vertices in state 1,*

$$\liminf_{t \rightarrow \infty} P(\xi_t(x) = 1) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} P(\xi_t(x) = 2) = 0 \quad \text{for all } x \in \mathbb{Z}^d.$$

The two parameter regions covered by Theorems 2 and 3 are illustrated in the last three phase diagrams of Figure 1. We point out that Theorem 3 holds even when $\lambda_{21} = \infty$, i.e., when the pathogen invades instantaneously nearby healthy hosts. Also, in addition to the statement of the theorem, our proof gives specific estimates on the rate of extinction of the pathogens and the rate of expansion of the healthy hosts. The first part of the proof shows that there exists a constant $c > 0$ such that, uniformly in all configurations ξ with infinitely many 1s and for any site x ,

$$P_\xi(\sup \{t : \xi_t(y) = 2 \text{ for some } y \text{ such that } |y - x| \leq e^{ct}\} < \infty) = 1 \quad (9)$$

where P_ξ refers to the law of the process starting from configuration ξ . In other words, there exists a uniform (over all sites) exponentially growing (in time) neighborhood of any site which is eventually void of pathogens. To describe the long-term behavior of the healthy hosts, let ζ_t denote the one-dimensional nearest-neighbor (supercritical) contact process with parameter λ_{10} starting from the all-one configuration. Also, let $\alpha > 0$ denote the edge speed in this contact process as defined in [4]. Then, under the assumptions of the theorem, there exist

- a random site X and an almost surely finite time T that depend on ξ ,
- site-valued processes $\ell_t \leq r_t$ defined for $t \geq T$ and satisfying $\ell_T = r_T = X$,
- a coupling of the processes ξ_t and ζ_t

such that, P_ξ -almost surely,

$$\begin{aligned} \lim_{t \rightarrow \infty} -\ell_t/t &= \lim_{t \rightarrow \infty} r_t/t = \alpha \\ \text{and } \xi_t(x) &= \zeta_t(x) \text{ for all } (x, t) \in [\ell_t, r_t] \times [T, \infty). \end{aligned} \quad (10)$$

In other words, as long as ξ has an infinite number of 1s then, P_ξ -almost surely, eventually there arises a stable population of 1s that behaves like the basic contact process on an interval that grows linearly in time. From (9)–(10), we also obtain a complete convergence theorem. Indeed, letting ν denote the upper invariant measure of the contact process ζ_t and δ_0 denote the measure that concentrates on the all-zero configuration, since the distribution of the contact process converges weakly to ν [11, Ch. VI], we deduce the following for the distribution μ_t of the process ξ_t .

Corollary 4 – Let $\lambda_{10} > \lambda_c > \lambda_{20}$ and $\delta = 0$ and $d = 1$. Then, as $t \rightarrow \infty$,

$$\mu_t \Rightarrow \rho \delta_0 + (1 - \rho) \nu \quad \text{where} \quad \rho = P_{\mu_0}(\{x : \xi_t(x) = 1\}) \neq \emptyset \text{ for all } t > 0).$$

In particular, all invariant measures are convex combinations of δ_0 and ν .

2. Mean-field model

In this section we prove Theorem 1. Part 1 is easily proved by examining $u_1 + u_2$. Adding the equations in (2) and recalling $u_0 = 1 - u_1 - u_2$, we find

$$\begin{aligned} (u_1 + u_2)' &= (\lambda_{10} u_1 + \lambda_{20} u_2)(1 - u_1 - u_2) - (u_1 + u_2) \\ &\leq \max(\lambda_{10}, \lambda_{20})(u_1 + u_2)(1 - u_1 - u_2) - (u_1 + u_2) \end{aligned}$$

therefore, if $\max(\lambda_{10}, \lambda_{20}) \leq 1$, then

$$(u_1 + u_2)' \leq -(u_1 + u_2)^2$$

which implies that $u_1 + u_2 \rightarrow 0$ as $t \rightarrow \infty$. Since $u_1, u_2 \geq 0$, this means that

$$u_1(t), u_2(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{for any value of} \quad u_1(0), u_2(0)$$

and part 1 is proved. To prove the other parts we need a few more ingredients.

Lemma 5 (Dulac's criterion) – Suppose we have the planar system

$$u' = F(u) \quad \text{where} \quad u \in \mathbb{R}^2 \quad \text{and} \quad F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is a } C^1 \text{ function.}$$

Let $R \subset \mathbb{R}^2$ be a simply connected region in the plane and $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 function (called a Dulac function) such that the divergence of BF is not zero and has constant sign on R . Then, the system $u' = F(u)$ has no closed orbits on R .

PROOF. This is a standard result. The proof uses Green's theorem and can be found in [13], Theorem 2 in Section 3.9. \square

Theorem 6 (Generalized Poincaré-Bendixson Theorem) – Let Λ be a compact invariant set of the planar system $u' = F(u)$ with $F \in C^1$, such that there is at most a finite number of equilibria in Λ . For $x \in \Lambda$, let $\omega(x)$ denote the ω -limit set of the positive semi-orbit starting from x . Then the following trichotomy holds:

1. $\omega(x)$ is an equilibrium,
2. $\omega(x)$ is a periodic orbit, or
3. $\omega(x)$ is the union of at least one equilibrium and at least one homoclinic or heteroclinic orbit, and any heteroclinic orbits form a single heteroclinic cycle connecting the equilibria in $\omega(x)$.

PROOF. This is Theorem 1.5 in [16], and is proved therein. \square

Lemma 7 – If Λ is a compact and simply connected invariant set of the planar system $u' = F(u)$ such that Dulac's criterion holds on Λ , and there are at most a finite number of equilibria in Λ , then for any $x \in \Lambda(x)$, $\omega(x)$ is an equilibrium.

PROOF. Under Dulac's criterion, in the trichotomy of Theorem 6, option 2 is ruled out, and homoclinic orbits are ruled out. Suppose $\omega(x)$ contains a heteroclinic orbit, then $\omega(x)$ contains at least two equilibria. This implies existence of a heteroclinic cycle connecting the equilibria in $\omega(x)$. But this is also ruled out by Dulac's criterion. \square

We now return to our model. Letting $u' = F(u)$ denote (2) and $B(u_1, u_2) = 1/(u_1 u_2)$,

$$\begin{aligned} \text{Div}(BF) &= \partial_{u_1}(\lambda_{10} u_0/u_2 - 1/u_2 - \lambda_{21} + \delta/u_1) \\ &\quad + \partial_{u_2}(\lambda_{20} u_0/u_1 - 1/u_1 + \lambda_{21} - \delta/u_1) \\ &= -\lambda_{10}/u_2 - \delta/u_1^2 - \lambda_{20}/u_1 \end{aligned}$$

is strictly negative on the interior of Λ , therefore B is a Dulac function for (2). Thus in order to prove the rest of Theorem 1, it is enough to identify all equilibria in Λ , then to show that all of them except p attract no points in Λ_+ . If this can be shown, then by Lemma 7, trajectories have no choice but to converge to p .

As noted in the Introduction, we have the three boundary equilibria, that we now label

1. $p_0 = (0, 0)$,
2. $p_1 = (1 - 1/\lambda_{10}, 0)$ when $\lambda_{10} > 1$ and
3. $p_2 = (0, 1 - 1/\lambda_{20})$ when $\delta = 0$ and $\lambda_{20} > 1$.

and these are the only equilibria on the boundary on Λ (note that $(u_1 + u_2)' < 0$ if $u_1 + u_2 = 1$). Simple conditions ensure instability of these equilibria.

Lemma 8 – *In each case, the stated equilibrium attracts no points in Λ_+ .*

1. p_0 , if $\lambda_{10} > 1$ or $\lambda_{20} > 1 + \delta$.
2. p_1 , if (3) holds.
3. p_2 , if (4) holds.

PROOF. It is easy to check that

1. If $\lambda_{10} > 1$ then $\partial_1 F_1(p_0) > 0$, and if $\lambda_{20} > 1 + \delta$ then $\partial_2 F_2(p_0) > 0$.
2. If (3) holds then $\partial_2 F_2(p_1) > 0$.
3. If (4) holds then $\partial_1 F_1(p_2) > 0$.

Letting p denote the relevant equilibrium, in each of the above cases, the result follows from the fact that

- $p \in \{u_i = 0\}$ and
- there exists a Λ_+ -neighborhood $U \subseteq \Lambda_+$ of p such that $F_i(u) > 0$ for all $u \in U$.

This completes the proof. \square

Next, we examine the nullclines. Write (2) as $u'_1 = F_1(u_1, u_2)$ and $u'_2 = F_2(u_1, u_2)$.

u_2 -nullcline. Solving $F_2(u_1, u_2) = 0$ gives the pair of conditions

$$(i) \ u_2 = 0 \quad \text{or} \quad (ii) \ \lambda_{20} u_0 - (1 + \delta) + \lambda_{21} u_1 = 0.$$

We distinguish three cases:

1. If $\lambda_{20} = \lambda_{21} = 0$ then (ii) is void.
2. If $\lambda_{20} = 0$ and $\lambda_{21} > 0$ then (ii) reads $u_1 = (1 + \delta)/\lambda_{21}$.
3. If $\lambda_{20} > 0$ then (ii) gives

$$u_2 = \left(-1 + \frac{\lambda_{21}}{\lambda_{20}}\right) u_1 + 1 - \frac{1 + \delta}{\lambda_{20}}.$$

In conclusion, the u_2 -nullcline is a pair of straight lines

$$\{u_2 = 0\} \quad \text{and} \quad \ell_2 = \begin{cases} \emptyset & \text{if } \lambda_{20} = \lambda_{21} = 0 \\ \{u_1 = (1 + \delta)/\lambda_{21}\} & \text{if } \lambda_{20} = 0 \text{ and } \lambda_{21} > 0 \\ \{u_2 = m_2 u_1 + b_2\} & \text{if } \lambda_{20} > 0 \end{cases}$$

where $m_2 = (-1 + \lambda_{21}/\lambda_{20})$ and $b_2 = 1 - (1 + \delta)/\lambda_{20}$.

u_1 -nullcline. Solving $F_1(u_1, u_2) = 0$ gives the quadratic

$$-\lambda_{10} u_1^2 - (\lambda_{10} + \lambda_{21}) u_1 u_2 + (\lambda_{10} - 1) u_1 + \delta u_2 = 0. \quad (11)$$

We distinguish two cases depending on the value of δ .

Case 1: $\delta = 0$. In this case, (11) factors to give the pair of conditions

$$(i) \ u_1 = 0 \quad \text{or} \quad (ii) \ -\lambda_{10} u_1 - (\lambda_{10} + \lambda_{21}) u_2 + (\lambda_{10} - 1) = 0.$$

We again distinguish three sub-cases:

1. If $\lambda_{10} = \lambda_{21} = 0$ then (ii) is void.
2. If $\lambda_{10} = 0$ and $\lambda_{21} > 0$ then (ii) gives $u_2 = -1/\lambda_{21}$ that does not intersect Λ .
3. If $\lambda_{10} > 0$ then (ii) gives

$$u_1 = -\left(1 + \frac{\lambda_{21}}{\lambda_{10}}\right) u_2 + 1 - \frac{1}{\lambda_{10}}.$$

In conclusion, the intersection of the u_1 -nullcline with Λ is a pair of straight lines

$$\{u_1 = 0\} \quad \text{and} \quad \ell_1 = \begin{cases} \emptyset & \text{if } \lambda_{10} \leq 1 \\ \{u_1 = m_1 u_2 + b_1\} & \text{if } \lambda_{10} > 1 \end{cases}$$

where $m_1 = -(1 + \lambda_{21}/\lambda_{10})$ and $b_1 = 1 - 1/\lambda_{10}$.

Case 2: $\delta > 0$. Let ℓ_1 denote the intersection of the u_1 -nullcline with R_+^2 , excluding the origin. There are a couple of possibilities.

1. If $\lambda_{10} = \lambda_{21} = 0$ then (11) degenerates to $u_2 = u_1/\delta$, so $\ell_1 = \{u_2 = u_1/\delta, u_1 > 0\}$.
2. If $\max(\lambda_{10}, \lambda_{21}) > 0$ then (11) is hyperbolic, which can be checked from the hyperbolicity condition $B^2 - 4AC > 0$ for a quadratic of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. Ignoring lower order terms gives

$$u_1 (\lambda_{10} u_1 + (\lambda_{10} + \lambda_{21}) u_2) = 0$$

which means there is a vertical asymptote and a slant asymptote of the form $u_1 = m_1 u_2 + b$ for the same m_1 as before, and some unknown b . Since $m_1 < 0$ and thus $1/m_1 < 0$, as a

function of u_1 , the slant asymptote has a negative slope. Ignoring the terms without u_2 , we find that the vertical asymptote is given by

$$-(\lambda_{10} + \lambda_{21})u_1 + \delta = 0.$$

Therefore, the vertical asymptote is $v := \{u_1 = \delta/(\lambda_{10} + \lambda_{21})\}$ and in particular, v lies to the right of the line $\{u_1 = 0\}$. Since $F_1(0, u_2) = \delta u_2$, the u_1 -nullcline cannot cross $\{u_1 = 0\}$ except possibly at p_0 . In addition,

$$F_1(u_1, 0) = 0 \quad \text{if and only if} \quad u_1 \in \{0, b_1\}$$

from which it follows that the u_1 -nullcline must intersect both p_0 and $(b_1, 0)$, and is otherwise disjoint from the straight line $\{u_2 = 0\}$. Combining these observations, the only option is that ℓ_1 is a single convex curve such that

- (a) if $b_1 \leq 0$ then ℓ_1 emerges from p_0 and asymptotes to v , and
- (b) if $b_1 > 0$ then ℓ_1 emerges from p_1 and asymptotes to v .

Note that if v intersects p_1 then $\ell_1 = v \cap \mathbb{R}_+^2$.

We are now ready to prove the theorem.

PROOF OF THEOREM 1.

Part 1. This was proved at the beginning of this section.

Part 2. Under any of the conditions in Part 2, from Lemma 8 it follows that no boundary equilibria attract any points in Λ_+ . Using Dulac's criterion and Lemma 7, any trajectory in Λ_+ converges to an equilibrium in Λ . Therefore, there must be at least one interior equilibrium. To obtain uniqueness, it suffices to show in each case that $\ell_1 \cap \ell_2$ is at most a single point.

Part 2a. Suppose $\delta > 0$, $\lambda_{10} \leq 1$ and $\lambda_{20} > 1 + \delta$. Then ℓ_2 is not vertical and intersects the line $\{u_1 = 0\}$ at the point p_2 that lies above p_0 . If $\lambda_{10} = \lambda_{21} = 0$ then ℓ_1 is a straight line that intersects p_0 , so $\ell_1 \cap \ell_2$ is at most a single point. Otherwise, ℓ_1 emerges from $(0, 0)$ into R_+^2 and asymptotes to v . Since ℓ_2 is a straight line and ℓ_1 is convex, a quick sketch confirms that the intersection $\ell_1 \cap \ell_2$ reduces to a single point.

Part 2b. Suppose $\delta > 0$, $\lambda_{10} > 1$ and (3) holds. Regardless of the position of v relative to p_1 , ℓ_1 is a convex (possibly straight) curve emerging from p_1 into R_+^2 . If ℓ_2 is vertical then, since

$$(1 + \delta)/\lambda_{21} > \delta/(\lambda_{10} + \lambda_{21}),$$

the asymptote v lies to the left of p_1 , so ℓ_1 is not vertical and $\ell_1 \cap \ell_2$ is at most a single point. If ℓ_2 is not vertical then, since it lies above p_1 , the intersection $\ell_1 \cap \ell_2$ reduces to a single point.

Part 2c. Suppose $\delta = 0$, $\lambda_{10} > 1$ and (3) holds. In addition, since $m_1 < 0$ and $b_1 > 0$, ℓ_1 intersects the ray $\{u_1 = 0, u_2 > 0\}$, and since condition (3) holds, either ℓ_2 lies to the left of p_1 if it is vertical or above p_1 if it is not vertical.

- Suppose $\lambda_{20} \leq 1$. If ℓ_2 is vertical then clearly $\ell_1 \cap \ell_2$ is a single point. If ℓ_2 is not vertical then, since it lies above p_1 and intersects $\{u_1 = 0\}$ below p_0 , we must have $m_2 > 0$, and again the intersection $\ell_1 \cap \ell_2$ reduces to a single point.

rate	symbol	effect on the process
$(\lambda_{10} - \lambda_{20})/2d$	$x \xrightarrow{1} y$	birth at y when x is occupied by a healthy host and y is empty
$\lambda_{20}/2d$	$x \xrightarrow{2} y$	birth at y when x is occupied and y is empty
$\lambda_{21}/2d$	$x \xrightarrow{3} y$	infection at y when x is infected and y is occupied by a healthy host
1	\times at x	death at x when x is occupied
δ	\bullet at x	recovery at x when x is infected

TABLE 1

Graphical representation of the process when $\lambda_{10} > \lambda_{20}$ (pathogen). The rates in the left column correspond to the different parameters of the independent Poisson processes, attached to either each oriented edge connected two neighbors (first three rows) or each vertex (last two rows).

- If $\lambda_{20} > 1$ and (4) holds, then ℓ_1 intersects $\{u_1 = 0\}$ above p_2 . Since ℓ_2 intersects $\{u_2 = 0\}$ above p_1 , ℓ_1 and ℓ_2 have different slopes, so $\ell_1 \cap \ell_2$ is again a single point.

Part 3. Suppose $\delta = 0$, $\lambda_{10} > 1$, (3) does not hold and (4) holds. Since $\lambda_{10} > 1$, p_0 attracts no points in Λ_+ , and if p_2 is an equilibrium then, since (4) holds, it attracts no points in Λ_+ . To conclude that all trajectories in Λ_+ converge to p_1 , it is enough to show there is no interior equilibrium. Since $\delta = 0$ and $\lambda_{10} > 0$, ℓ_1 is a straight line with finite slope as a function of u_1 . If ℓ_2 is vertical then, since (3) does not hold, it either intersects or lies to the right of p_1 , in which case there is no interior equilibrium. If ℓ_2 is not vertical and $\lambda_{20} \leq 1$ then, since $b_2 \leq 0$ and (3) does not hold, on $\{u_1 > 0\}$, ℓ_2 lies either on or below the line $\{u_2 = 0\}$ and there is no interior equilibrium. If $\lambda_{20} > 1$ then, since (4) holds, ℓ_1 intersects $\{u_1 = 0\}$ above p_2 , so on Λ_+ , ℓ_2 lies below ℓ_1 and there is no interior equilibrium.

Part 4. Suppose $\delta = 0$, $\lambda_{20} > 1$, (3) does not hold and (4) holds. Essentially the same argument as in Part 3, with the roles of type 1 and 2 exchanged, shows that p_2 attracts all points in Λ_+ .

The proof is complete. \square

3. Graphical representation

Throughout the paper, we think of the process as being generated from a percolation sub-structure, also called Harris' graphical representation [9]. That is, we attach independent Poisson processes with appropriate rate to each vertex and oriented edge of the d -dimensional integer lattice. The process is then constructed by assuming that, at the times of these Poisson processes, either a birth or an infection or a death or a recovery occurs whenever the configuration of the system at that time is compatible with the event. Table 1 shows how to construct the percolation substructure when $\lambda_{10} > \lambda_{20}$, in which case the symbiont is a pathogen.

Note that the results in (5)–(7) can be proved by coupling different processes using this graphical representation rather than Theorem III.1.5 in [11]. For instance, the contact process ζ_t^2 with parameter λ_{20} can be constructed from the graphical representation in the table by assuming that births can only occur through type 2 arrows, while the contact process ζ_t^1 with parameter λ_{10} can be constructed by assuming that births occur through both type 1 and type 2 arrows. Constructing our modified stacked contact process and these two contact processes from this common graphical representation results in a coupling such that

$$\{x \in \mathbb{Z}^d : \zeta_t^2(x) \neq 0\} \subset \{x \in \mathbb{Z}^d : \xi_t(x) \neq 0\} \subset \{x \in \mathbb{Z}^d : \zeta_t^1(x) \neq 0\}$$

at all times t provided this holds at time zero. This shows that (5) when $\lambda_{10} > \lambda_{20}$. This property when the inequality is reversed as well as (6) and (7) are proved similarly by using other graphical representations which are designed based on the ordering of the parameters.

4. Proof of Theorem 2

The background idea to prove Theorem 2 is a block construction, which consists in coupling the process suitably rescaled in space and time with oriented site percolation. This technique has been introduced in [2] and is reviewed in [5]. Even though it has been widely used in the field of interacting particle systems, finding an appropriate coupling is rarely straightforward because it strongly depends on the process under consideration. To prove the theorem, we need a series of estimates, starting with some geometrical properties of the configurations.

For simplicity, we focus on the two-dimensional case but our proof easily extends to higher dimensions. Note however that the proof relies on geometrical properties which are not true for the one-dimensional system. To begin with, we let N be a large integer to be fixed later and partition the two-dimensional lattice using the collection of boxes

$$B_z := 2zN + \{-N, \dots, N-2, N-1\}^2 \quad \text{for all } z \in \mathbb{Z}^2. \quad (12)$$

Letting $e_1 := (1, 0)$ be the first unit vector, the first step of the proof is to show that if the number of symbionts in the spatial region B_0 is not too small while the number of symbionts in B_{e_1} is not too large, then there exists a connected component void of symbionts that contains many vertices in the region B_{e_1} and whose boundary contains many symbionts. To make this statement precise, for every configuration ξ of the system, we introduce

$$\begin{aligned} S_z(\xi) &:= \{x \in B_z : \xi(x) = 2\} = \text{subset of the box } B_z \text{ occupied by symbionts} \\ C_1(\xi) &:= \text{largest connected component of } B_{e_1} \setminus S_{e_1}(\xi) \\ C_0(\xi) &:= \text{connected component of } (B_0 \cup B_{e_1}) \setminus (S_0(\xi) \cup S_{e_1}(\xi)) \text{ containing } C_1(\xi) \\ B_0(\xi) &:= \{x \in B_0 \cup B_{e_1} : \xi(x) = 2 \text{ and } y \in C_0(\xi) \text{ for some } y \sim x\} \\ &= \text{number of symbionts along the boundary of the cluster } C_0(\xi). \end{aligned}$$

Then, we have the following lemma.

Lemma 9 – Assume that $\text{card}(S_0(\xi)) = K_0$ and $\text{card}(S_{e_1}(\xi)) = K_1 \leq N$. Then,

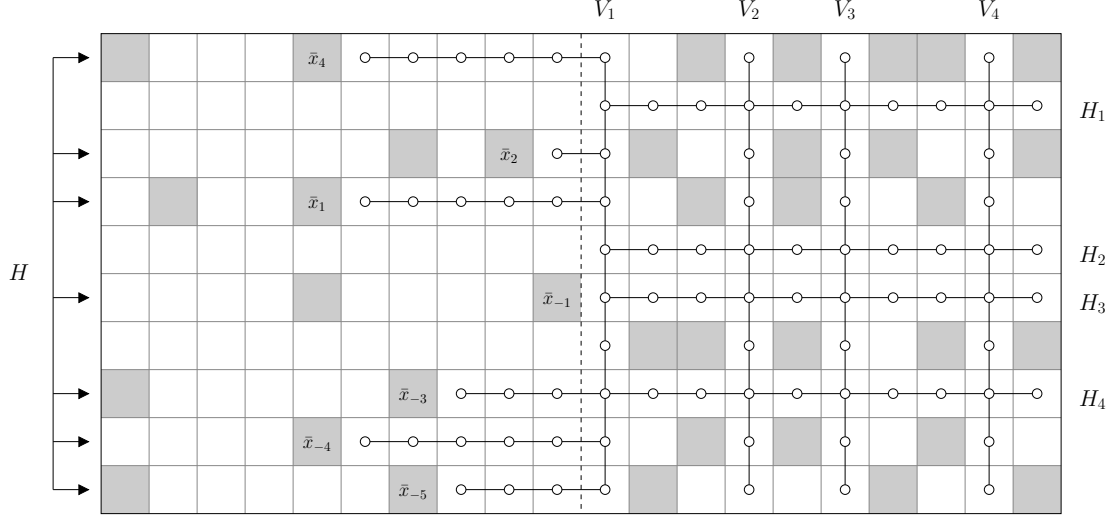
$$\text{card}(C_1(\xi)) \geq (2N)^2 - K_1^2 \geq 3N^2 \quad \text{and} \quad \text{card}(B_0(\xi)) \geq \sqrt{K_0}.$$

PROOF. To prove the first inequality, we first observe that at most K_1 rows in B_{e_1} can be occupied by a symbiont so at least $2N - K_1$ rows have no symbiont. Reasoning similarly with the columns, we deduce that there exist coordinates

$$N \leq x_1 < \dots < x_{2N-K_1} < 3N \quad \text{and} \quad -N \leq y_1 < \dots < y_{2N-K_1} < N$$

such that $\xi((x, y)) \neq 2$ for all vertices (x, y) belonging to

$$\begin{aligned} V_i &:= \{(x, y) \in B_{e_1} : x = x_i\} \quad \text{for some } i = 1, 2, \dots, 2N - K_1 \\ \text{or } H_j &:= \{(x, y) \in B_{e_1} : y = y_j\} \quad \text{for some } j = 1, 2, \dots, 2N - K_1. \end{aligned}$$

FIGURE 2. Construction in Lemma 9 when $K_H \geq K_V$.

Since each of the columns V_i intersects each of the rows H_j , the union of all these sets forms a connected set. In addition, the number of vertices in this connected set exceeds half of the number of vertices in the box B_{e_1} . More precisely,

$$\begin{aligned} \Gamma &:= V_1 \cup \dots \cup V_{2N-K_1} \cup H_1 \cup \dots \cup H_{2N-K_1} \text{ is connected} \\ \text{and } \text{card}(\Gamma) &= (2N)^2 - K_1^2 \geq 4N^2 - N^2 = 3N^2 > (1/2)(2N)^2. \end{aligned}$$

The right-hand side of Figures 2–3, in which the grey squares represent the symbionts, shows a picture of Γ in white dots. In particular, any connected component of the set of vertices not occupied by a symbiont that does not contain Γ has less vertices than Γ . It directly follows that

$$\Gamma \subset C_1(\xi) \quad \text{and} \quad \text{card}(C_1(\xi)) \geq \text{card}(\Gamma) = (2N)^2 - K_1^2 \geq 3N^2.$$

To prove the second inequality, we let

$$\begin{aligned} H &:= \{y \in [-N, N] : \xi((x, y)) = 2 \text{ for some } x \in [-N, N]\} \\ V &:= \{x \in [-N, N] : \xi((x, y)) = 2 \text{ for some } y \in [-N, N]\} \end{aligned}$$

be the coordinates of the rows, respectively, the coordinates of the columns, in B_0 that have at least one symbiont. When there are K_0 symbionts in B_0 , we must have

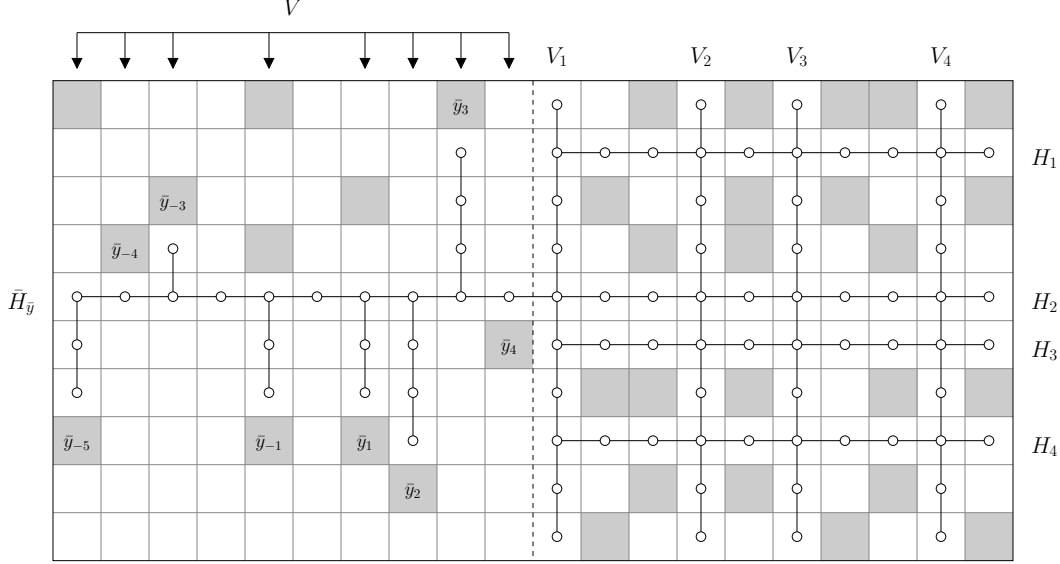
$$(a) \ K_H \geq K_V \text{ and } K_H \geq \sqrt{K_0} \quad \text{or} \quad (b) \ K_V > K_H \text{ and } K_V \geq \sqrt{K_0}$$

where $K_H := \text{card}(H)$ and $K_V := \text{card}(V)$. Assume (a) and let

$$\bar{x}_y = \max \{x \in [-N, x_1] : \xi((x, y)) = 2\} \quad \text{for all } y \in H.$$

The horizontal coordinates \bar{x}_y for all $y \in H$ are shown in the rightmost grey squares on the left-hand side of Figure 2. By construction, we have

$$\xi((\bar{x}_y, y)) = 2 \quad \text{and} \quad \xi((\bar{x}_y + 1, y)), \xi((\bar{x}_y + 2, y)), \dots, \xi((x_1, y)) \neq 2.$$

FIGURE 3. Construction in Lemma 9 when $K_V > K_H$.

Since in addition $V_1 \subset \Gamma \subset C_1(\xi) \subset C_0(\xi)$, we have $(\bar{x}_y, y) \in B_0(\xi)$ therefore

$$\text{card}(B_0(\xi)) \geq \text{card}\{(\bar{x}_y, y) : y \in H\} = \text{card}(H) = K_H \geq \sqrt{K_0}.$$

Now, assume (b). In this case,

- since $K_H < K_V \leq 2N$, there exists $\bar{y} \in [-N, N)$ such that

$$\bar{H}_{\bar{y}} := \{(x, \bar{y}) : x \in [-N, N)\} \text{ has no symbiont,}$$

- for each $x \in V$, there exists \bar{y}_x such that (x, \bar{y}_x) is occupied by a symbiont and has a neighbor connected to (x, \bar{y}) by a vertical path of vertices not occupied by a symbiont.

A particular choice of the vertical coordinates \bar{y}_x for $x \in V$ and the vertical paths to $\bar{H}_{\bar{y}}$ are shown on the left-hand side of Figure 3. Since in addition each vertex in $\bar{H}_{\bar{y}}$ is connected by a path of vertices not occupied by a symbiont to each vertex in V_1 , we deduce that

$$\bar{H}_{\bar{y}} \subset C_0(\xi) \quad \text{and} \quad (x, \bar{y}_x) \in B_0(\xi) \quad \text{for all} \quad x \in V.$$

In particular, we conclude as previously that

$$\text{card}(B_0(\xi)) \geq \text{card}\{(x, \bar{y}_x) : x \in V\} = \text{card}(V) = K_V \geq \sqrt{K_0}.$$

This completes the proof. \square

Using Lemma 9, we now prove that if the number of symbionts in B_0 is not too small and the number of symbionts in B_{e_1} not too large then, with probability close to one, one of the infected hosts along the boundary of the largest connected component void of symbionts will recover after a short time. This is done in Lemma 10. After this recovery event, with probability close to one,

healthy hosts quickly invade the connected component void of symbionts when their natural birth rate is sufficiently large. This provides a habitat for the symbionts which, again with probability close to one, quickly invade the connected component in a short time when the infection rate is sufficiently large. This creates in particular a large cluster of symbionts in the target region B_{e_1} , as shown in Lemma 11. To quantify these statements, we set

$$\begin{aligned} H^-(z, k) &:= \{\text{card}(S_z(\xi_t)) \leq k \text{ for all } t \in (0, 1)\} \\ H^+(z, k) &:= \{\text{card}(S_z(\xi_t)) \geq k \text{ for all } t \in (0, 1)\}. \end{aligned} \quad (13)$$

Also, we introduce the stopping time

$$T_0 := \inf \{t : \xi_t(x) = 1 \text{ for some } x \in B_0(\xi_{t-})\}.$$

In the next lemma, we give an upper bound for

$$P(T_0 > 1/2 \mid H^+(0, N^{1/4}) \cap H^-(e_1, N)). \quad (14)$$

We point out that our bound holds regardless of the configuration in B_{e_1} and therefore without conditioning on the event $H^-(e_1, N)$. The reason for the extra conditioning is that the conditional probability in (14) is the one that appears naturally in the proof of Lemma 13 below. The two events in the conditioning are convenient because they give information about the number of symbionts at *all* times until time 1 rather than at a fixed time but we will prove that, with probability close to one when the parameter N is large,

- if there are at least N symbionts at some time before time 1 in the box B_{e_1} , which is the event $(H^-(e_1, N))^c$, then there are at least \sqrt{N} symbionts in this box at time 1,
- if there are at least \sqrt{N} symbionts at time 0 in box B_0 then there are at least $N^{1/4}$ symbionts in this box at all times until time 1, which is the event $H^+(0, 0)$.

This will be used to couple the process with oriented site percolation.

Lemma 10 – *For all $N > 0$, we have*

$$P(T_0 > 1/2 \mid H^+(0, N^{1/4}) \cap H^-(e_1, N)) \leq \exp(-(\delta/2) N^{1/8}).$$

PROOF. According to the second part of Lemma 9,

$$\begin{aligned} &H^+(0, N^{1/4}) \cap H^-(e_1, N) \\ &= \{\text{card}(S_0(\xi_t)) \geq N^{1/4} \text{ and } \text{card}(S_{e_1}(\xi_t)) \leq N \text{ for all } t \in (0, 1)\} \\ &\subset \{\text{card}(B_0(\xi_t)) \geq N^{1/8} \text{ for all } t \in (0, 1)\}. \end{aligned}$$

Since in addition infected hosts recover at rate δ ,

$$\begin{aligned} &P(T_0 > 1/2 \mid H^+(0, N^{1/4}) \cap H^-(e_1, N)) \\ &\leq P(\text{Exponential}(\delta N^{1/8}) > 1/2) = \exp(-(\delta/2) N^{1/8}), \end{aligned}$$

which completes the proof. \square

Lemma 11 – Assume that $\min(\lambda_{21}, \lambda_{10}/\lambda_{21}) \geq N^6$. Then,

$$P(T_0 \leq 1/2 \text{ and } H^-(e_1, N) \mid H^+(0, N^{1/4})) \leq 8N^2 \exp(-N/8) + 8(3 + \delta)/N$$

for all N sufficiently large.

PROOF. First, we define the stopping times

$$\begin{aligned} T_1 &:= \inf \{t > T_0 : \xi_t(x) = 1 \text{ for all } x \in C_0(\xi_{T_0})\} \\ T_2 &:= \inf \{t > T_1 : \xi_t(x) = 2 \text{ for all } x \in C_0(\xi_{T_0})\}. \end{aligned}$$

Also, for $j = 1, 2, \dots, 8N^2$, we let

$$X_j = \text{Exponential}(\lambda_{10}) \quad \text{and} \quad X = \text{Exponential}(8(1 + \delta + \lambda_{20} + \lambda_{21})N^2)$$

be independent. Since both the connected set $C_0(\xi_{T_0})$ and its boundary have at most $8N^2$ vertices, the conditional probability given $T_0 \leq 1/2$ that this connected set becomes fully occupied by healthy hosts before time $3/4$ is bounded from below by the probability that $8N^2$ consecutive births of healthy hosts occur in less than $1/4$ unit of time and before any birth of infected hosts, death, infection or recovery occurs. This gives the lower bound:

$$\begin{aligned} P(T_1 \leq 3/4 \mid T_0 \leq 1/2 \text{ and } H^+(0, N^{1/4})) \\ &\geq P(X_1 + X_2 + \dots + X_{8N^2} < \min(1/4, X)) \\ &\geq P(X_1 + X_2 + \dots + X_{8N^2} < (\lambda_{21}N^3)^{-1}) P(X > (\lambda_{21}N^3)^{-1}) \\ &\geq P(X_j < (8\lambda_{21}N^5)^{-1} \text{ for } j = 1, 2, \dots, 8N^2) P(X > (\lambda_{21}N^3)^{-1}) \\ &= (1 - \exp(-(\lambda_{10}/8\lambda_{21}N^5)))^{8N^2} \exp(-8(1 + \delta + \lambda_{20} + \lambda_{21})/\lambda_{21}N) \\ &\geq (1 - 8N^2 \exp(-(\lambda_{10}/8\lambda_{21}N^5)))(1 - 8(1 + \delta + \lambda_{20} + \lambda_{21})/\lambda_{21}N) \\ &\geq 1 - 8N^2 \exp(-(\lambda_{10}/8\lambda_{21}N^5)) - 8(1 + \delta + \lambda_{20} + \lambda_{21})/\lambda_{21}N \\ &\geq 1 - 8N^2 \exp(-N/8) - 16/N \end{aligned} \tag{15}$$

for all N large. Note that, in the last inequality, we have used that

$$1 + \delta + \lambda_{20} \leq \lambda_{21} \quad \text{whenever} \quad \lambda_{21} \geq N^6 \text{ and } N \text{ is large.}$$

Using the same argument but for infections rather than births of healthy hosts, we obtain that the conditional probability that $C_0(\xi_{T_0})$ becomes fully infected before time 1 is

$$\begin{aligned} P(T_2 \leq 1 \mid T_0 \leq 1/2 \text{ and } T_1 \leq 3/4 \text{ and } H^+(0, N^{1/4})) \\ &\geq (1 - \exp(-(\lambda_{21}/8N^5)))^{8N^2} \exp(-8(1 + \delta)/N) \\ &\geq (1 - 8N^2 \exp(-(\lambda_{21}/8N^5)))(1 - 8(1 + \delta)/N) \\ &\geq 1 - 8N^2 \exp(-(\lambda_{21}/8N^5)) - 8(1 + \delta)/N \\ &\geq 1 - 8N^2 \exp(-N/8) - 8(1 + \delta)/N \end{aligned} \tag{16}$$

for all N large. Combining (15)–(16), we deduce that

$$\begin{aligned}
P(\xi_t \not\equiv 2 \text{ on } C_0(\xi_{T_0}) \text{ for all } t \in (T_0, 1) \mid T_0 \leq 1/2 \text{ and } H^+(0, N^{1/4})) \\
\leq P(T_2 > 1 \mid T_0 \leq 1/2 \text{ and } H^+(0, N^{1/4})) \\
\leq P(T_1 > 3/4 \mid T_0 \leq 1/2 \text{ and } H^+(0, N^{1/4})) \\
\quad + P(T_2 > 1 \mid T_0 \leq 1/2 \text{ and } T_1 \leq 3/4 \text{ and } H^+(0, N^{1/4})) \\
\leq 8N^2 \exp(-N/8) + 8(3 + \delta)/N.
\end{aligned} \tag{17}$$

In other respects, according to the first part of Lemma 9,

$$\begin{aligned}
H^+(0, N^{1/4}) \cap H^-(e_1, N) \\
= \{\text{card}(S_0(\xi_t)) \geq N^{1/4} \text{ and } \text{card}(S_{e_1}(\xi_t)) \leq N \text{ for all } t \in (0, 1)\} \\
\subset \{\text{card}(C_1(\xi_t)) \geq 3N^2 \text{ for all } t \in (0, 1)\}.
\end{aligned}$$

Since in addition $C_1(\xi_{T_0}) = C_0(\xi_{T_0}) \cap B_{e_1}$, it follows that the two events

$$\begin{aligned}
H^+(0, N^{1/4}) \cap H^-(e_1, N) \quad \text{and} \\
\Omega := \{\xi_t \equiv 2 \text{ on } C_0(\xi_{T_0}) \text{ for some } t \in (T_0, 1)\}
\end{aligned}$$

are mutually exclusive. This, together with (17), implies that

$$\begin{aligned}
P(T_0 \leq 1/2 \text{ and } H^-(e_1, N) \mid H^+(0, N^{1/4})) \\
= P(T_0 \leq 1/2 \text{ and } H^-(e_1, N) \cap \Omega^c \mid H^+(0, N^{1/4})) \\
\leq P(T_0 \leq 1/2 \text{ and } \Omega^c \mid H^+(0, N^{1/4})) \\
\leq P(\Omega^c \mid T_0 \leq 1/2 \text{ and } H^+(0, N^{1/4})) \leq 8N^2 \exp(-N/8) + 8(3 + \delta)/N
\end{aligned}$$

for all N sufficiently large. \square

In order to ultimately compare the process with oriented site percolation, the next step is to attach a collection of events to the collection of boxes defined in (12). Declare site

$$(z, n) \in \mathcal{H} := \{(z, n) \in \mathbb{Z}^2 \times \mathbb{N} : z_1 + z_2 + n \text{ is even}\}$$

to be good when the event

$$H^*(z, n) := \{\text{card}(S_z(\xi_n)) \geq \sqrt{N}\} \tag{18}$$

that the region B_z contains at least \sqrt{N} symbionts at time n occurs. To prove that the set of good sites dominates the set of wet sites in an oriented site percolation process on \mathcal{H} , we need to relate the events (13) to the good event (18). More precisely, we need to prove that both

$$P(H^*(e_1, 1) \mid (H^-(e_1, N))^c) \quad \text{and} \quad P(H^+(0, N^{1/4}) \mid H^*(0, 0))$$

are close to one, which follows from the next lemma.

Lemma 12 – For all $z \in \mathbb{Z}^2$ and $K \leq N^2$,

$$\begin{aligned}
P(\text{card}(S_z(\xi_t)) \leq (K/2) e^{-(1+\delta)} \text{ for some} \\
t \in (0, 1) \mid \text{card}(S_z(\xi_0)) = K) \leq \exp(-K e^{-(4+\delta)})
\end{aligned}$$

uniformly in all possible configurations outside B_z .

PROOF. Since the total rate at which a host associated with a symbiont either dies with its symbiont or recovers is equal to $1 + \delta$, we have

$$P(\xi_t(x) = 2 \text{ for all } t \in (0, 1) \mid \xi_0(x) = 2) = e^{-(1+\delta)}.$$

By independence, we deduce that, conditional on $\text{card}(S_z(\xi_0)) = K$,

$$\text{card}\{x \in B_z : \xi_t(x) = 2 \text{ for all } t \in (0, 1)\} = \text{Binomial}(K, e^{-(1+\delta)}) \quad (19)$$

in distribution. Using the large deviation estimate

$$P(\text{Binomial}(K, p) \leq K(p - z)) \leq \exp(-Kz^2/2p) \quad \text{for all } z \in (0, p)$$

with $p = e^{-(1+\delta)}$ and $z = p/2$, together with (19), we conclude that

$$\begin{aligned} P(\text{card}(S_z(\xi_t)) \leq (K/2) e^{-(1+\delta)} \text{ for some } t \in (0, 1) \mid \text{card}(S_z(\xi_0)) = K) \\ \leq P(\text{card}\{x \in B_z : \xi_t(x) = 2 \text{ for all } t \in (0, 1)\} \leq (K/2) e^{-(1+\delta)} \mid \text{card}(S_z(\xi_0)) = K) \\ \leq P(\text{Binomial}(K, e^{-(1+\delta)}) \leq (K/2) e^{-(1+\delta)}) \\ \leq \exp(-Kp/8) \leq \exp(-Ke^{-(4+\delta)}). \end{aligned}$$

This completes the proof. \square

With Lemmas 10–12 in hands, we are now ready to prove that if a site is good then its neighbors at the next level are good with probability arbitrarily close to one.

Lemma 13 – *For all $\epsilon > 0$, there exists N large such that*

$$P((H^*(e_1, 1))^c \mid H^*(0, 0)) \leq \epsilon \quad \text{whenever} \quad \min(\lambda_{21}, \lambda_{10}/\lambda_{21}) \geq N^6.$$

PROOF. The conditional probability to be estimated is bounded by

$$\begin{aligned} P((H^*(e_1, 1))^c \mid H^*(0, 0)) \\ \leq P((H^*(e_1, 1))^c \mid (H^-(e_1, N))^c \cap H^+(0, N^{1/4}) \cap H^*(0, 0)) \\ + P(H^-(e_1, N) \mid H^+(0, N^{1/4}) \cap H^*(0, 0)) \\ + P((H^+(0, N^{1/4}))^c \mid H^*(0, 0)). \end{aligned} \quad (20)$$

The first and last terms on the right-hand side of (20) can be bounded using Lemma 12. Indeed, according to this lemma, we have

$$\begin{aligned} P((H^*(e_1, 1))^c \mid (H^-(e_1, N))^c \cap H^+(0, N^{1/4}) \cap H^*(0, 0)) \\ = P(\text{card}(S_{e_1}(\xi_1)) < \sqrt{N} \mid \text{card}(S_{e_1}(\xi_t)) > N \\ \text{for some } t \in (0, 1) \text{ and } H^+(0, N^{1/4}) \cap H^*(0, 0)) \\ \leq P(\text{card}(S_{e_1}(\xi_1)) \leq (N/2) e^{-(1+\delta)} \mid \text{card}(S_{e_1}(\xi_t)) > N \\ \text{for some } t \in (0, 1) \text{ and } H^+(0, N^{1/4}) \cap H^*(0, 0)) \\ \leq \exp(-N e^{-(4+\delta)}) \leq \epsilon/3 \end{aligned} \quad (21)$$

for all N large. Lemma 12 also implies that

$$\begin{aligned}
& P((H^+(0, N^{1/4}))^c \mid H^*(0, 0)) \\
&= P(\text{card}(S_0(\xi_t)) < N^{1/4} \text{ for some } t \in (0, 1) \mid \text{card}(S_0(\xi_0)) \geq \sqrt{N}) \\
&\leq P(\text{card}(S_0(\xi_t)) < (\sqrt{N}/2) e^{-(1+\delta)} \text{ for some } t \in (0, 1) \mid \\
&\quad \text{card}(S_0(\xi_0)) \geq \sqrt{N}) \leq \exp(-\sqrt{N} e^{-(4+\delta)}) \leq \epsilon/3
\end{aligned} \tag{22}$$

for all N large. In addition, by Lemmas 10–11, for all N large,

$$\begin{aligned}
& P(H^-(e_1, N) \mid H^+(0, N^{1/4}) \cap H^*(0, 0)) \\
&= P(T_0 > 1/2 \text{ and } H^-(e_1, N) \mid H^+(0, N^{1/4}) \cap H^*(0, 0)) \\
&\quad + P(T_0 \leq 1/2 \text{ and } H^-(e_1, N) \mid H^+(0, N^{1/4}) \cap H^*(0, 0)) \\
&\leq P(T_0 > 1/2 \mid H^+(0, N^{1/4}) \cap H^-(e_1, N) \cap H^*(0, 0)) \\
&\quad + P(T_0 \leq 1/2 \text{ and } H^-(e_1, N) \mid H^+(0, N^{1/4}) \cap H^*(0, 0)) \\
&\leq \exp(-(\delta/2) N^{1/8}) + 8N^2 \exp(-N/8) + 8(3 + \delta)/N \leq \epsilon/3
\end{aligned} \tag{23}$$

when $\min(\lambda_{21}, \lambda_{10}/\lambda_{21}) \geq N^6$. Combining (20)–(23), we conclude that

$$P((H^*(e_1, 1))^c \mid H^*(0, 0)) \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

for all N sufficiently large. \square

Lemma 13, together with [5, Theorem 4.3] and obvious symmetry, implies that, at least for N large, the set of good sites dominates stochastically the set of wet sites in a one-dependent oriented site percolation process on \mathcal{H} in which sites are closed with probability at most ϵ . Since such a process is supercritical, i.e., percolation occurs with probability one when starting with a positive density of open sites at level zero, and since $\epsilon > 0$ in the statement of the lemma can be chosen arbitrarily small, we deduce that the density of boxes with at least \sqrt{N} symbionts converges to a positive limit. This shows survival of the symbionts, and therefore coexistence of the hosts with their symbionts, under the assumptions of Theorem 2.

5. Proof of Theorem 3

This section is devoted to the proof of (9)–(10) which, together, imply Theorem 2. As pointed out in the introduction, the theorem is true even in the limiting case $\lambda_{21} = \infty$, i.e., when the pathogen invades instantaneously the nearby healthy hosts. Note that, in this case, if

$$\xi_{t-}(x) = 2, \quad \xi_{t-}(x+1) = 0, \quad \xi_{t-}(x+2) = \cdots = \xi_{t-}(x+k) = 1$$

and a $0 \rightarrow 1$ birth event occurs along the edge $(x+2, x+1)$ at time t then

$$\xi_{t-}(x) = \xi_{t-}(x+1) = \xi_{t-}(x+2) = \cdots = \xi_{t-}(x+k) = 2.$$

The same applies in the reflected scenario; we refer to this as *invasion*. For convenience's sake, in the limiting case $\lambda_{21} = \infty$, we ignore the set of configurations

$$\{\xi : \xi(x) = 2 \text{ and } \xi(y) = 1 \text{ for some } x, y \text{ such that } |x - y| = 1\}$$

since the complement of this set is invariant under the dynamics.

Theorem 3 is proved in several steps, but the basic recipe is to focus on some collection of 2s until they collide with the 1s (or on some collection of 1s, until they collide with the 2s), then to reset and try again. In the absence of collision, things are nice, so if we can show that the time to collision is short and the probability of collision is less than one, then we need only reset a finite number of times before we can say something about the asymptotics. For the sake of exposition, we first study the “segregated” case where the 1s and the 2s lie on separate sides. We then treat the general case in the same way as the segregated case, using a simple comparison property.

Segregated Case. Given a configuration ξ_t , we let

$$r_t := \sup \{x \in \mathbb{Z} : \xi_t(x) = 2\} \quad \text{and} \quad \ell_t := \inf \{x \in \mathbb{Z} : \xi_t(x) = 1\}$$

be the rightmost type 2 and the leftmost type 1, respectively. Say that configuration ξ_t is *segregated* if either $r_t < \ell_t$ or this is true after reflecting across the origin. Since $\delta = 0$, if interactions are nearest-neighbor then segregation is preserved, that is,

$$\xi_0 \text{ segregated} \quad \text{implies that} \quad \xi_t \text{ segregated for all } t > 0.$$

Letting $d_t := \ell_t - r_t$, another way to express this is: $d_0 \geq 1$ implies that $d_t \geq 1$ for all $t > 0$. For a segregated initial configuration, and noting that d_t is left skip-free, let

$$\tau := \inf \{t > 0 : d_{t-} = 2 \text{ and } d_t = 1\} \quad \text{when} \quad \lambda_{21} < \infty$$

be the first time a 1 and a 2 are adjacent after separation. When the infection rate is infinite, d_t is never equal to one, so in this case we instead define

$$\tau := \inf \{t > 0 : d_{t-} = 2 \text{ and } r_t > r_{t-}\} \quad \text{when} \quad \lambda_{21} = \infty$$

since this implies that either a 1 showed up, and was instantaneously invaded by a 2, or else a 2 advanced and then invaded some 1s at time t . In Lemmas 14–15 below, we show that there are positive constants p, c and C so that uniformly over all segregated configuration ξ ,

$$P_\xi(\tau = \infty) \geq p, \quad P_\xi(r_t > r_0 - e^{ct} \mid \tau = \infty) \leq C e^{-ct}, \quad P_\xi(t < \tau < \infty) \leq C e^{-ct}.$$

Thinking of the event that $\tau = \infty$ as a success, the first estimate says that there is a success with positive probability and the last estimate that, if they happen, failures happen quickly, which will be used later to prove that the first success occurs quickly as well. The second estimate indicates that, in case of a success, the pathogens die exponentially fast. Using these estimates, we can then prove the upcoming Propositions 16 and 17, that are the main results in the segregated case. In the following proofs, c and C are strictly positive constants such that $c, 1/C$ decrease from step to step. To check that decreasing $c, 1/C$ will be justified, note that the probability estimates used are upper bounds, and that except at a few specific points when c is fixed while C is varied, the events being measured do not increase when $c, 1/C$ are decreased.

Lemma 14 – *There is $p > 0$ so that*

$$P_\xi(\tau = \infty) \geq p \quad \text{uniformly in all segregated } \xi.$$

Also, there are $c, C > 0$ so that, for all segregated ξ and all $t > 0$,

$$P_\xi(r_s > r_0 - e^{cs} \text{ for some } s > t \mid \tau = \infty) \leq C e^{-ct}.$$

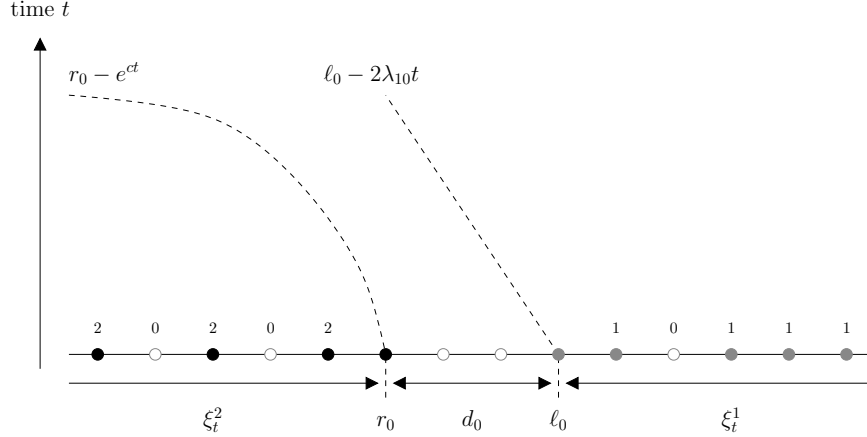


FIGURE 4. Picture related to the proof of Lemma 14

PROOF. For an illustration of some of the random variables and processes introduced in the proof, we refer the reader to Figure 4. Using a pair of independent substructures, define a pair of contact processes ζ_t^1 and ζ_t^2 with respective birth rates λ_{10} and λ_{20} and initial configurations

$$\zeta_0^2(x) = \mathbf{1}\{x \in \mathbb{Z} : \xi_0(x) = 2\} \quad \text{and} \quad \zeta_0^1(x) = \mathbf{1}\{x \in \mathbb{Z} : \xi_0(x) = 1\}.$$

There is a natural coupling of our process ξ_t with the two contact processes ζ_t^1 and ζ_t^2 obtained by defining ξ_t using both substructures up to time τ , then using, say, only the first substructure for all times $t > \tau$. This coupling has the property that

$$\{x : \xi_t(x) = 2\} = \{x : \zeta_t^2(x) = 1\} \quad \text{and} \quad \{x : \xi_t(x) = 1\} = \{x : \zeta_t^1(x) = 1\}$$

for all $t \leq \tau$. Now, define

$$r_t^2 = \sup \{x \in \mathbb{Z} : \zeta_t^2(x) = 1\} \quad \text{and} \quad \ell_t^1 = \inf \{x \in \mathbb{Z} : \zeta_t^1(x) = 1\},$$

so that $r_t = r_t^2$ and $\ell_t = \ell_t^1$ for all $t \leq \tau$. By the coupling,

$$\{\tau < \infty\} = \{\ell_t^1 - r_t^2 = 1 \text{ for some } t \geq 0\} \tag{24}$$

and we point out that the right-hand side of (24) can be estimated in terms of the two independent contact processes ζ_t^1 and ζ_t^2 , which is the key to proving the first estimate.

In the estimates that follow, c, C and D represent universal, positive constants and c, C may get smaller, respectively, larger from step to step. By monotonicity of the contact process, for $i = 1, 2$, the set of occupied sites $\{x : \zeta_t^i(x) = 1\}$ is dominated by the pure birth process in which particles do not die and give birth onto neighboring sites at rate λ_{i0} , so the advance of type i into uncharted territory grows like at most Poisson $(\lambda_{i0} t)$. In particular,

$$P(\ell_t^1 - r_t^2 < n) \leq P(d_0 - \text{Poisson}((\lambda_{10} + \lambda_{20})t) < n) \quad \text{for all } n > 0,$$

and applying a standard large deviations estimate, we get

$$P(\ell_s^1 - r_s^2 < 2 \text{ for some } s \leq d_0/(2(\lambda_{10} + \lambda_{20}))) \leq C e^{-cd_0}. \tag{25}$$

Also, for each $t > 0$,

$$P(\ell_s^1 < \ell_0 - 2\lambda_{10}t \text{ for some } s \leq t) \leq e^{-ct}. \quad (26)$$

To control r_t^2 , we use a known estimate at integer times, then a Poisson estimate at in-between times. From [8] and the assumption $\lambda_{20} < \lambda_c$, for any integer n ,

$$P(r_n^2 > r_0 - e^{cn}) \leq C e^{-cn}. \quad (27)$$

In addition, since the displacement in one unit of time is dominated by a Poisson random variable with parameter λ_{20} , for any integer $k \geq 0$, we have

$$P(r_s^2 - r_n^2 > k \text{ for some } s \in [n, n+1]) \leq C e^{-ck}. \quad (28)$$

Combining (27) and (28), we deduce that

$$P(r_s^2 > r_0 - e^{cn} + n \text{ for some } s \in [n, n+1]) \leq C e^{-cn} \quad (29)$$

Then, combining with (26) evaluated at $t = n+1$, for each integer $n \geq 1$,

$$P(\ell_s^1 - r_s^2 < d_0 + e^{cn} - (1 + 2\lambda_{10})(n+1) \text{ for some } s \in [n, n+1]) \leq C e^{-cn}.$$

To deduce the first estimate, we distinguish two cases:

Case 1. In case $d_0 > D$, the above bound on $\ell_s^1 - r_s^2$ is at least two for all n . Combining the observation (24) with (25) and (29) and summing over $n \geq \lfloor d_0/(2(\lambda_{10} + \lambda_{20})) \rfloor$,

$$\begin{aligned} P(\tau < \infty) &= P(\ell_t^1 - r_t^2 = 1 \text{ for some } t \geq 0) \\ &\leq e^{-cd_0} + \sum_{n \geq cd_0} C e^{-cn} \leq C e^{-cd_0} \quad \text{when } d_0 > D. \end{aligned} \quad (30)$$

Case 2. Assume now that $d_0 \leq D$. In this case,

$$P(\ell_1^1 - r_1^2 > D) = q > 0 \quad \text{when } \zeta_0^1 = \mathbf{1}\{x \leq 0\} \text{ and } \zeta_0^2 = \mathbf{1}\{x \geq 2\}.$$

By monotonicity and translation invariance, the same holds in general:

$$P(\ell_1^1 - r_1^2 > D) \geq q > 0 \quad \text{when } \ell_0^1 - r_0^2 > 1.$$

Using the Markov property to combine with (30) gives the first part of the lemma.

Now, in (29) above, for $n \geq n_0$ for some n_0 , absorb n into $-e^{cn}$ by decreasing c , then increase C to account for $n < n_0$. Then, for any $t > 0$, summing (29) over $n \geq \lfloor t \rfloor$,

$$P(r_s^2 > r_0^2 - e^{-cs} \text{ for some } s > t) \leq C e^{-ct}. \quad (31)$$

On the event $\{\tau = \infty\}$, we have $r_t = r_t^2$ for all $t \geq 0$. Using this and (31),

$$\begin{aligned} P_\xi(r_s > r_0 - e^{cs} \text{ for some } s > t \mid \tau = \infty) \\ &= P_\xi(r_s > r_0 - e^{cs} \text{ for some } s > t \text{ and } \tau = \infty) / P_\xi(\tau = \infty) \\ &\leq P_\xi(r_s^2 > r_0 - e^{cs} \text{ for some } s > t) / P_\xi(\tau = \infty) \leq (C/p) e^{-ct} \leq C e^{-ct} \end{aligned}$$

uniformly in all segregated ξ . \square

Next, we use estimates from the preceding proof to control $P_\xi(t < \tau < \infty)$.

Lemma 15 – *There are positive constants c and C so that*

$$P_\xi(t < \tau < \infty) \leq C e^{-ct} \quad \text{uniformly in all segregated } \xi.$$

PROOF. Define recursively $\sigma_1 = \tau$ and

$$\sigma_{i+1} = \inf \{t > \tau_i : \ell_{t-}^1 - r_{t-}^2 = 2 \text{ and } \ell_t^1 - r_t^2 = 1\} \quad \text{for all } i > 0.$$

We will use the obvious inclusion

$$\begin{aligned} \{t < \tau < \infty\} &= \{t < \sigma_1 < \infty\} \\ &\subset \{t < \sigma_i < \infty \text{ for some } i > 0\} = \{\ell_s^1 - r_s^2 = 1 \text{ for some } s > t\} \end{aligned}$$

because the probability of the event on the right-hand side is easier to estimate. Using the two bounds (26) and (27) above, uniformly in all configurations with $\ell_0^1 - r_0^2 \geq 1$,

$$P(\ell_n^1 - r_n^2 < 2 + e^{cn} - 2\lambda_{10}n) \leq C e^{-cn}$$

and for n large enough the above bound on $\ell_n^1 - r_n^2$ is at least n (it is exponential in n but this is good enough). For $n = \lfloor t \rfloor$ with t large enough and using (30),

$$\begin{aligned} P(t < \tau < \infty) &= P(t < \sigma_1 < \infty) \leq P(t < \sigma_i < \infty \text{ for some } i) \\ &\leq P(\ell_n^1 - r_n^2 < n) + P(n < \sigma_i < \infty \text{ for some } i \mid \ell_n^1 - r_n^2 \geq n) \\ &\leq C e^{-cn} + C e^{-cn} \leq C e^{-ct}. \end{aligned}$$

This completes the proof. \square

In the next proposition, we use the estimates of the previous two lemmas to prove that, when starting from a segregated configuration, the position of the rightmost pathogen goes to $-\infty$ exponentially fast. This is the analog of the second estimate in Lemma 14 but without the conditioning. This result is then used in the subsequent lemma to show that the probability that the rightmost pathogen moves n steps to the right of its initial position decays exponentially with n .

Proposition 16 – *There are positive constants c and $C > 0$ so that*

$$P_\xi(r_t > r_0 - e^{ct}) \leq C e^{-ct} \quad \text{uniformly in all segregated } \xi.$$

PROOF. Define recursively $\tau_0 = 0$, $\tau_1 = \tau$ and

$$\tau_{i+1} = \inf \{t > \tau_i : d_{t-} = 2 \text{ and } d_t = 1\} \quad \text{for all } i > 0.$$

Then, by Lemma 14,

$$N := \sup \{i : \tau_i < \infty\} \preceq -1 + \text{Geometric}(p)$$

where \preceq means stochastically smaller. Moreover, by Lemma 15,

$$(\tau_i - \tau_{i-1}) \mathbf{1}\{N \geq i\} \preceq T_i \mathbf{1}\{N \geq i\} \quad \text{for all } i > 0$$

where T_1, T_2, \dots is a sequence of independent, identically distributed random variables independent of the random variable N and such that

$$P(T_i > t) = \min(Ce^{-ct}, 1) \quad \text{for all } t > 0.$$

Therefore, letting $n = \lceil t/(2E(T_i)) \rceil$ and applying a large deviations bound,

$$\begin{aligned} P(\tau_N > t) &\leq P(T_1 + \dots + T_N > t) \\ &\leq P(N > n) + P(T_1 + \dots + T_n > 2n E(T_i)) \leq C e^{-cn} \leq C e^{-ct}. \end{aligned}$$

We distinguish two cases depending on the value of the infection rate.

Case 1. Assume first that $\lambda_{21} < \infty$. Recall that $\lambda_{20} \leq \lambda_{21}$ by assumption. Comparing the set of sites in state 2 to a pure birth process with no deaths and with birth to adjacent sites at rate λ_{21} , a large deviations estimate gives $c, C > 0$ so that

$$P(r_t > r_0 + 2\lambda_{21}t) \leq C e^{-ct} \quad \text{for all } t > 0. \quad (32)$$

For any $c > 0$, there is t_0 so that $\lambda_{21}t < e^{ct} - e^{ct/2}$ for all $t > t_0$. Therefore,

$$\begin{aligned} P(r_t > r_0 - e^{ct/2}) &\leq P(r_{t/2} > r_0 + \lambda_{21}t) \\ &\quad + P(r_t > r_0 - e^{ct/2} \text{ and } r_{t/2} \leq r_0 + \lambda_{21}t) \\ &\leq P(r_{t/2} > r_0 + \lambda_{21}t) + P(r_t > r_{t/2} - \lambda_{21}t - e^{ct/2}) \\ &\leq P(r_{t/2} > r_0 + \lambda_{21}t) + P(r_t > r_{t/2} - e^{ct}) \end{aligned} \quad (33)$$

for all $t > t_0$. In other respects,

$$\begin{aligned} P(r_t > r_{t/2} - e^{ct}) &\leq P(t/2 < \tau_N < \infty) \\ &\quad + P(r_{t/2} > r_0 - e^{ct} \mid \tau = \infty) \leq C e^{-ct}. \end{aligned} \quad (34)$$

Combining equations (32)–(34) gives the desired estimate when $t > t_0$. If $t \leq t_0$ then, after increasing C if necessary, the estimate holds for all values of t .

Case 2. Now, let $\lambda_{21} = \infty$. By assumption, $d_0 > 1$. We have

$$r_t = r_t - r_{\tau_{N_t}} + \sum_{i=1}^{N_t} (r_{\tau_i} - r_{\tau_i^-}) + (r_{\tau_i^-} - r_{\tau_{i-1}}) \quad \text{where } N_t = \sup \{i : \tau_i < t\}.$$

For time $t \notin \{\tau_i : i \geq 0\}$, the process r_t grows at rate at most λ_{20} , so repeating the reasoning in case 1 with the rate λ_{20} replacing the rate λ_{21} , we get

$$P\left(r_t - r_{\tau_{N_t}} + \sum_{i=1}^{N_t} (r_{\tau_i^-} - r_{\tau_{i-1}}) > -e^{ct}\right) \leq C e^{-ct}. \quad (35)$$

It remains to control the jumps $r_{\tau_i} - r_{\tau_i^-}$. Note that, for all $i = 1, 2, \dots, N_t$,

$$\tau_i - \tau_{i-1} \succeq T_i = \text{Exponential}(\lambda_{10} + \lambda_{20})$$

where the random variables T_i are also independent. A straightforward coupling argument shows that, for any i , the set $\{x \geq \ell_{\tau_i} : \xi_t(x) = 1\}$ is dominated by product measure with density $u(t)$ satisfying the initial-value problem

$$\frac{d}{dt} u(t) = -u(t) + 2\lambda_{10}(1 - u(t)) \quad \text{with } u(0) = 1.$$

Letting $u^* = 2\lambda_{10}/(1 + 2\lambda_{10})$, the unique solution is

$$u(t) = u^* + (1 - u^*) e^{-(1+2\lambda_{10})t}.$$

Conditioned on $\tau_i = t$,

$$r_t - r_{t-} \preceq X(t) = \text{Geometric}(u(t)).$$

Bounding $u(t)$ by $1 - ct$ on $[0, 1]$, using the fact that $X(t)$ is stochastically decreasing in t to integrate against $T_i = \exp(\lambda_{10} + \lambda_{20})$ in place of τ , then using the bound $dP(T_i = t) \leq C$,

$$\begin{aligned} P(r_{\tau_i} - r_{\tau_i^-} > x) &= \int_0^\infty P(r_t - r_{t-} > x \mid \tau_i = t) dP(\tau_i = t) \\ &\leq C \int_0^1 (1 - ct)^{\lfloor x \rfloor} dt + (1 - c)^{\lfloor x \rfloor} \leq C/x. \end{aligned} \quad (36)$$

Letting $n = \lceil 2(\lambda_{10} + \lambda_{20})t \rceil$, a large deviations estimate gives

$$P(T_1 + T_2 + \dots + T_n < t) \leq C e^{-ct}.$$

For the same n , and using (36) with $x = e^{ct/2}$,

$$\begin{aligned} P\left(\sum_{i=1}^{N_t} (r_{\tau_i} - r_{\tau_i^-}) > e^{ct/2}\right) &\leq P\left(\sum_{i=1}^n T_i < t\right) + P\left(\sum_{i=1}^n r_{\tau_i^+} - r_{\tau_i} > e^{ct/2}\right) \\ &\leq C e^{-ct} + \sum_{i=1}^n P(r_{\tau_i^+} - r_{\tau_i} > e^{ct/2}) \leq C e^{-ct} + Cn e^{-ct/2} \leq C e^{-ct}. \end{aligned} \quad (37)$$

Combining (37) with (35), then noticing that $e^{ct} - e^{ct/2} > e^{ct/2}$ when $t > t_0$ for some t_0 that depends on c , and increasing C if necessary to account for $t < t_0$, we have the desired result:

$$\begin{aligned} P(r_t > r_0 - e^{ct/2}) &\leq P\left(r_t - r_{\tau_{N_t}} + \sum_{i=1}^{N_t} (r_{\tau_i} - r_{\tau_{i-1}}) > -e^{ct}\right) \\ &\quad + P\left(\sum_{i=1}^{N_t} (r_{\tau_i} - r_{\tau_i^-}) > e^{ct/2}\right) \leq C e^{-ct}. \end{aligned}$$

This completes the proof. \square

Proposition 17 – *There is $C > 0$ so that*

$$P_\xi(r_t > r_0 + n \text{ for some } t \geq 0) \leq C e^{-cn} \quad \text{uniformly in all segregated } \xi.$$

PROOF. We again distinguish two cases.

Case 1. Assume that $\lambda_{21} < \infty$. Comparing to a pure birth process as above,

$$P(r_s > r_0 + n \text{ for some } s \leq m_0) \leq C e^{-cn} \quad \text{for } m_0 = \lfloor n/2\lambda_{21} \rfloor.$$

For all n large enough, $-e^{cm} + m < n$ for all m , therefore using Proposition 16 and large deviations for the Poisson distribution with parameter λ_{21} ,

$$\begin{aligned} P(r_s > r_0 + n \text{ for some } s \in [m, m+1]) \\ \leq P(r_m > r_0 - e^{-cm}) + P(r_s > r_m + m \text{ for some } s \in [m, m+1]) \leq C e^{-cm}. \end{aligned}$$

Summing over $m \geq m_0$ gives the desired estimate.

Case 2. Now, assume that $\lambda_{21} = \infty$. The idea is the same as above, and the complications are the same as in the proof of Proposition 16, namely, that we need to account for invasion – the only difference is that on a fixed interval of time $[m, m+1]$ we can have as many as a Poisson (λ_{10}) number of invasions. We omit the details. \square

General Case. We now use the result proved in the segregated case to study the process starting from a general initial configuration. To deal with this general context, it is useful to have a notion of *descendant* and *ancestor*. Suppose $\xi_s(x) = \xi_t(y) = i$ for some x, y and $s \leq t$. Then (y, t) is a descendant of (x, s) , and (x, s) is an ancestor of (y, t) if there are times and sites

$$s = t_0 < t_1 < \dots < t_{k-1} \leq t_k = t \quad \text{and} \quad x = x_1, x_2, \dots, x_k = y$$

such that the following two conditions hold:

- For $j = 1, 2, \dots, k$, we have $\xi_r(x_j) = i$ for all times $r \in [t_{j-1}, t_j]$.
- For $j = 1, 2, \dots, k-1$, we have $\xi_{t_j^-}(x_{j+1}) \neq \xi_{t_j}(x_{j+1}) = i$ as a result of a $0 \rightarrow i$ event along the edge (x_j, x_{j+1}) at time t_j .

In case $\lambda_{21} = \infty$, there may be a chain of multiple sites at a fixed time, but the reader may verify that the ancestry is still well-defined, in the obvious way.

For a site x , let A_t^x denote the descendants of $(x, 0)$, which all have type $\xi_0(x)$. By default, the set of descendants is empty if site x is initially in state 0. Since interactions are nearest-neighbor, it is easy to check that

$$\xi_0(x) = 1 \quad \text{implies that} \quad \xi_t(y) \neq 2 \quad \text{for all } y \in [\inf A_t^x, \sup A_t^x] \quad \text{and } t \geq 0,$$

and similarly with the roles of states 1 and 2 reversed.

Proposition 18 – *There are $c, C > 0$ such that*

$$P_\xi(A_t^x \neq \emptyset) \leq C e^{-ct} \quad \text{and} \quad P_\xi(A_t^x \not\subset [x-n, x+n] \text{ for some } t \geq 0) \leq C e^{-cn}$$

uniformly in all configurations ξ such that $\xi(x) = 2$.

PROOF. Let ξ_t^1 and ξ_t^2 be two copies of our modified stacked contact process constructed from the same graphical representation and define

$$\phi_t^1(x) = \mathbf{1} \{ \xi_t^1(x) \neq 0 \} \quad \text{and} \quad \phi_t^2(x) = \mathbf{1} \{ \xi_t^2(x) \neq 0 \}.$$

In case $\xi_0^2(x) = \phi_0^1(x) = \mathbf{1} \{ \xi_0^1(x) \neq 0 \}$, since $\lambda_{20} \leq \lambda_{10}$, by comparing to a (single-type) contact process with parameter λ_{10} , it is clear that

$$\phi_t^1(x) \leq \phi_t^2(x) \quad \text{for all } (x, t) \in \mathbb{Z} \times \mathbb{R}_+.$$

With this comparison property in hand, the proof follows that of Lemmas 14 and 15 and Propositions 16 and 17, so we only point out the adjustments.

Let $\ell_t = \inf A_t^x$ and $r_t = \sup A_t^x$ as well as

$$a_t = \sup \{ y < \ell_t : \xi_t(y) \neq 0 \} \quad \text{and} \quad b_t = \inf \{ y > r_t : \xi_t(y) \neq 0 \}.$$

Proposition 19 – For a site $x \in \mathbb{Z}$, let

$$\ell_t^x = \sup \{y \leq x : \xi_t(y) = 2\} \quad \text{and} \quad r_t^x = \inf \{y \geq x : \xi_t(y) = 2\}.$$

Then, there exist $c, C > 0$ such that, for any ξ and t_0 ,

$$P_\xi(\ell_t^x > x - e^{ct} \text{ or } r_t^x < x + e^{ct} \text{ for some } t > t_0) \leq C e^{-ct_0}.$$

Also, there exists $c > 0$ so that, for any configuration ξ and any site x ,

$$P_\xi(\sup \{t : \xi_t(y) = 2 \text{ for some } y \text{ such that } |y - x| \leq e^{ct}\} < \infty) = 1.$$

PROOF. Throughout this proof, y refers to a site which is initially in state 2. Let c, C be two constants as in Proposition 18, so that

$$P_\xi(A_n^y \neq \emptyset) \leq C e^{-cn} \quad \text{for all } y, \xi \text{ and } n.$$

Using a union bound over $y \in [x - e^{cn/2}, x + e^{cn/2}]$ and that $A_t^y = \emptyset$ is an absorbing property,

$$P_\xi(A_t^y \neq \emptyset \text{ for some } y \text{ such that } \xi_0(y) = 2 \\ \text{and } |y - x| \leq e^{cn/2} \text{ and some } t \geq n) \leq C e^{-cn/2}.$$

Let n_0 be such that $n \geq n_0$ implies $e^{cn/2} - e^{c(n+1)/4} > n$. If $|y - x| = \lceil e^{cn/2} \rceil + m$ for integer $m \geq 0$ and all $n \geq n_0$, then we have

$$P_\xi(|\sup A_t^y - x| \leq e^{c(n+1)/4} \text{ for some } t \geq 0) \leq C e^{-c(n+m)}$$

Taking C larger if necessary makes the previous inequality true also for all $n < n_0$. Then, taking a union bound and increasing C at the last step gives

$$P_\xi(|\sup_s \sup \xi_s^y - x| \leq e^{c(n+1)/4} \text{ for some } y \\ \text{such that } |y - x| > e^{cn/2} \text{ and } \xi_0(y) = 2) \leq C(1 - e^{-c}) e^{-cn} = C e^{-cn}.$$

Combining the estimates gives

$$P_\xi(\xi_t(y) = 2 \text{ for some } y \in [x - e^{ct/4}, x + e^{ct/4}] \text{ and } t \in [n, n+1]) \leq C e^{-cn/2}.$$

Given t_0 , summing over $n \geq \lfloor t_0 \rfloor$ gives the first statement. Summing over all n and using the Borel-Cantelli lemma finishes the proof. \square

From now on, we call an *event* a Poisson event in the graphical representation. Events are referred to by their effect on the target site, so for example, a $0 \rightarrow 1$ event is an edge event at time t along a directed edge (x, y) such that, if $\xi_{t-}(x) = 1$ and $\xi_{t-}(y) = 0$ then $\xi_t(y) = 1$. Also, a death event refers to a $\star \rightarrow 0$ event, while a birth event is a $0 \rightarrow \star$ event, where $\star \neq 0$. Note that since the graphical representation consists of at most a countably infinite number of Poisson point processes, with probability one, no two events occur at the same time.

Lemma 20 – For any $\mu > 0$, there are $c, C > 0$ so that

$$P_\xi(\xi_t(x) = 2 \text{ for some } |x| \leq n/2 + \mu t) \leq C e^{-cn}$$

for all ξ with $\xi(x) \neq 2$ and all $|x| \leq n$.

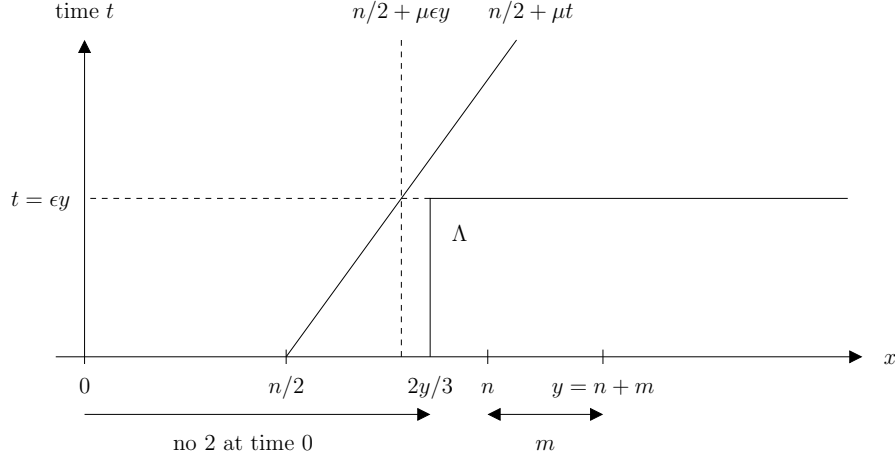


FIGURE 6. Picture related to the proof of Lemma 3

PROOF. Let $y = n + m$ with $m > 0$; a similar argument applies to $-y$. Using Proposition 18 gives the existence of constants $c, C > 0$ so that, for all $\epsilon > 0$,

$$P(A_{\epsilon y}^y \neq \emptyset \text{ or } \inf_t A_t^y < 2y/3) \leq C e^{-c\epsilon y}.$$

On the complement of the above event, the descendants of $(y, 0)$ are contained in

$$\Lambda := \{(x, t) : x \geq 2y/3 \text{ and } 0 \leq t \leq \epsilon y\}.$$

A quick sketch (see Figure 6) shows that the rectangle Λ is disjoint from the set

$$\{(x, t) : t \geq 0 \text{ and } |x| \leq n/2 + \mu t\}$$

provided the top left corner of Λ lies to the right of the line $x = n/2 + \mu t$, which is the condition

$$n/2 + \mu\epsilon y < 2y/3.$$

Since $n \leq y$, this condition is satisfied if $\epsilon < 1/6\mu$. Summing over $m > 0$ for both $y = n + m$ and $y = -n - m$ then gives the desired result. \square

A related notion to the descendants is the *cluster*, that we need only define for type 1, as follows. Suppose $\xi_s(x) = \xi_t(y) = 1$ for some x, y and $s \leq t$. Then, we say that (y, t) belongs to the cluster of (x, s) if there are times and sites

$$s = t_0 < t_1 < \dots < t_{k-1} \leq t_k = t \quad \text{and} \quad x = x_1, x_2, \dots, x_k = y$$

such that the following two conditions hold:

- For $j = 1, 2, \dots, k$, we have $\xi_r(x_j) = 1$ for all times $r \in [t_{j-1}, t_j]$.
- For $j = 1, 2, \dots, k-1$, there is a $0 \rightarrow 1$ birth event along the edge (x_j, x_{j+1}) at time t_j .

In contrast to the definition of descendants, it is permitted to have $\xi_{t_j^-}(x_{j+1}) = 1$.

If $\xi_s(x) = 1$ then, for $t \geq s$, let $B_t(x, s)$ denote the cluster of (x, s) at time t , that is,

$$B_t(x, s) := \{y \in \mathbb{Z} : (y, t) \text{ is in the cluster of } (x, s)\},$$

and denote it $B_t(x)$ for $s = 0$. Again, since interactions are nearest-neighbor,

$$\xi_t(y) \neq 2 \quad \text{for all } y \in [\inf B_t(x, s), \sup B_t(x, s)] \quad \text{and} \quad t \geq s.$$

As a warm-up to (10), we prove the following.

Lemma 21 – Let $\ell_t = \inf B_t(0)$ and $r_t = \sup B_t(0)$. For $\lambda_{21} < \infty$, let

$$\tau = \inf \{t > 0 : \xi_t(\ell_t - 1) = 2 \text{ or } \xi_t(r_t + 1) = 2\}$$

and for $\lambda_{21} = \infty$ define τ in a similar way. Then, there are $p, c, C > 0$ such that

$$P_\xi(\tau = \infty) \geq p \quad \text{and} \quad P_\xi(t < \tau < \infty) \leq C e^{-ct}$$

uniformly in all ξ such that $\xi(0) = 1$.

PROOF. Define a pair of independent copies ξ_t^1 and ξ_t^2 of the modified stacked contact process with initial configurations

$$\xi_0^1(y) = \xi_0(y) \mathbf{1}\{y = 0\} \quad \text{and} \quad \xi_0^2(y) = \xi_0(y) \mathbf{1}\{y \neq 0\}.$$

Define also

$$\begin{aligned} \ell_t^1 &= \inf \{x : \xi_t^1(x) = 1\} \quad \text{and} \quad r_t^1 = \sup \{x : \xi_t^1(x) = 1\} \\ a_t^2 &= \sup \{x < \ell_t^1 : \xi_t^2(x) = 2\} \quad \text{and} \quad b_t^2 = \inf \{x > r_t^1 : \xi_t^2(x) = 2\} \end{aligned}$$

so that τ can also be expressed as

$$\tau = \inf \{t > 0 : \ell_t^1 - a_t^2 \leq 1 \text{ or } b_t^2 - r_t^1 \leq 1\}.$$

To show that $\tau = \infty$ with positive probability, first we fix n and consider the case $\min(|a_0^2|, |b_0^2|) \geq n$. Using large deviations estimates for the Poisson distribution, we can show that

$$P(\max(|\ell_t^1|, |r_t^1|) \geq 2\lambda_{10}t + n/2 - 1 \text{ for some } t > 0) \leq C e^{-cn}.$$

To do so, it suffices to first make an estimate for $t \leq m_0 := \lfloor n/4\lambda_{10} \rfloor$, then for $t \in [m, m+1]$ for each $m \geq m_0$, then to take a union bound. Then, taking $\mu = 2\lambda_{10}$ and the same n as in the statement of Lemma 20, we find that

$$P(\min(|a_t^2|, |b_t^2|) \leq 2\lambda_{10}t + n/2 \text{ for some } t > 0) \leq C e^{-cn}.$$

Since in addition

$$\min(|a_t^2|, |b_t^2|) > \max(|\ell_t^1|, |r_t^1|) \text{ for all } t > 0 \quad \text{implies that} \quad \tau = \infty,$$

taking n large enough, we find that if $\xi(x) \neq 2$ for $|x| \leq n$ then

$$P_\xi(\tau = \infty) \geq 1/2.$$

For ξ such that $\xi(0) = 1$, the probability

$$P_\xi(\xi_1(0) = 1 \text{ and } \xi_1(x) = 0 \text{ for all } 0 < |x| \leq n)$$

is at least the probability that, on the time interval $[0, 1]$, there are no birth events along edges touching $[-n, n]$, there is no death event at 0, and there is a death event at every x with $0 < |x| \leq n$, and this probability is at least $2p$ for some $p > 0$. Using the Markov property and the estimate on $\tau = \infty$ in the previous case then gives $P_\xi(\tau = \infty) \geq p > 0$ as desired.

To deduce the estimate on $P_\xi(t < \tau < \infty)$, we note that

$$P(\max(|\ell_s^1|, |r_s^1|) \geq 2\lambda_{10} s \text{ for some } s \geq t) \leq C e^{-ct}$$

which can be proved by applying an estimate at each integer time $n > t$ and summing over n . Then, combining with the first statement in Proposition 19 and noting that

$$t < \tau < \infty \text{ implies that } \max(|\ell_s^1|, |r_s^1|) \geq \min(|a_s^2|, |b_s^2|) - 1 \text{ for some } s > t,$$

we deduce the estimate on $P_\xi(t < \tau < \infty)$. \square

We are now ready to establish (10) which states the existence of a linearly growing region starting from a random space-time point in which the process agrees with the contact process with parameter λ_{10} . This will also complete the proof of Theorem 3.

PROOF OF (10). Given (x, s) , for $t \geq s$ recall that $B_t(x, s)$ denotes the cluster of (x, s) at time t . Let $\tau_0 = 0$ and x_0 be any site with $\xi(x_0) = 1$. Without loss of generality, suppose that the set $\{x > 0 : \xi(x) = 1\}$ is infinite, and define x_i and τ_i recursively for $\lambda_{21} < \infty$ by letting

$$\begin{aligned} \ell_t^i &= \inf B_t(x_i, \tau_i) \\ r_t^i &= \sup B_t(x_i, \tau_i) \\ \tau_{i+1} &= \inf \{t > \tau_i : \xi_t(\ell_t^i - 1) = 2 \text{ or } \xi_t(r_t^i + 1) = 2\} \\ x_{i+1} &= \inf \{x > x_i : \xi_{\tau_{i+1}}(x) = 1\} \end{aligned}$$

with the value of x_i being unimportant if $\tau_i = \infty$. Note that if time $\tau_i < \infty$ then site x_{i+1} is well-defined due to the fact that

$$\{\xi : \xi(x) = 1 \text{ for infinitely many } x > 0\}$$

is an invariant set for the dynamics. Let $N = \sup \{i : \tau_i < \infty\}$. Applying the strong Markov property and using Lemma 21, we obtain that N is at most geometric with parameter p . In addition, by the second part of Lemma 21, for $i = 0, 1, 2, \dots$,

$$\tau_{i+1} - \tau_i \preceq T_i \quad \text{where} \quad P(T_i > t) \leq \max(1, C e^{-ct})$$

and the random variables T_i are independent. In particular, τ_N is almost surely finite. Let $T = \tau_N$ and let $X = x_N$, and let $\ell_t = \ell_t^N$ and $r_t = r_t^N$.

Recall that ζ_t denotes the process with initial configuration $\zeta_0(x) = 1$ for all x . Since $\lambda_{10} > \lambda_{20}$ by assumption, a straightforward coupling argument shows that, for any configuration ξ_0 ,

$$\{x \in \mathbb{Z} : \xi_t(x) \neq 0\} \subseteq \{x \in \mathbb{Z} : \zeta_t(x) = 1\}.$$

Therefore, $\zeta_{\tau_i}(x_i) = 1$ whenever $\tau_i < \infty$. By definition of time τ_{i+1} , the set $B_t(x_i, \tau_i)$ is the set of infected sites in a (single-type) contact process started from the single infected site x_i at time τ_i , so a coupling of [4] shows that if $\tau_i < \infty$ then

$$\xi_t(x) = \zeta_t(x) \text{ for all } x \in [\ell_t^i, r_t^i] \text{ and all } \tau_i < t < \tau_{i+1}.$$

In [4], it is shown that, for the contact process (which in this context means in the absence of any interaction with 2s), $-\ell_t^i/t$ and $r_t^i/t \rightarrow \alpha > 0$ so the same is true here provided $\tau_{i+1} = \infty$, which is the case for $i = N$. The proof is now complete. \square

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